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Cohomogeneity-one Einstein–Weyl structures: a local approach

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Abstract

We analyse in a systematic way the (non-) compact n-dimensional Einstein–Weyl spaces equipped with a cohomogeneity-one metric. In that context, with no compactness hypothesis for the manifold on which lives the Einstein–Weyl structure, we prove that, as soon as the $(n-1)$-dimensional space is a homogeneous reductive Riemannian space with a unimodular group of left-acting isometries G:

- there exists a Gauduchon gauge such that the Weyl-form is co-closed and its dual is a Killing vector for the metric;
- in that gauge, a simple constraint on the conformal scalar curvature holds;
- a non-exact Einstein–Weyl structure may exist only if the $(n-1)$-dimensional homogeneous space G/H contains a non-trivial subgroup H' that commutes with the isotropy subgroup H;
- the extra isometry due to this Killing vector corresponds to the right-action of one of the generators of the algebra of the subgroup H'.

The first two results are well known when the Einstein–Weyl structure lives on a compact manifold, but our analysis gives the first hints on the enlargement of the symmetry due to the Einstein–Weyl constraint.

We also prove that the subclass with G compact, a one-dimensional subgroup H' and the $(n-2)$-dimensional space $G/(H \times H')$ being an arbitrary compact symmetric Kähler coset space, corresponds to n-dimensional Riemannian locally conformally Kähler metrics. The explicit family of structures of cohomogeneity-one under $SU(n/2)$ being, thanks to our results, invariant under $U(1) \times SU(n/2)$, it coincides with the one first studied by Madsen; our analysis allows us to prove most of his conjectures.

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1. Introduction

For the last 30 years, gauge invariance has been the guiding idea in the construction of an unified theory of all interactions. In the genesis of the "gauge principle", the name of the mathematician Hermann Weyl should be recognised by physicists (the book of O'Raifeartaigh [1] offers a very instructive historical review of that subject). The same mathematician has also defined "Weyl geometry" which emphasises the role of conformal invariance: it describes not a given metric g in the target space together with a gauge field γ_μ (or a one-form $\gamma = \gamma_\mu \, \mathrm{d}x^\mu$), but an equivalence class $[g]$, through a conformal transformation of the distance $g \to e^f g$ and a related gauge transformation of the gauge field $\gamma \to \gamma + \mathrm{d}f$. So, even if H. Weyl's original hope [1–4] for an unified theory of electromagnetism and gravity failed, it is useful to pursue some analyses of his geometry (see some recent efforts in the same spirit in [5–7]). On the other hand, Einstein manifolds enter the game with Einstein gravity and also, since 1969 [8–10], in the framework of the quantisation of non-linear σ models: indeed, they offer multiplicatively renormalisable two-dimensional theories. Note that special Einstein manifolds are the Ricci-flat ones, e.g. the Calabi–Yau manifolds, the building block of string theory. It is then natural to export such Einstein constraints on a Weyl space, i.e. to study Einstein–Weyl geometry (for a recent review, see [11] and references therein).

Then, Einstein–Weyl geometry — in particular in three- and four-dimensions — has raised some interest in the last years, mainly among mathematicians, but also for physicists when three-dimensional Einstein–Weyl geometries were used to construct four-dimensional non-linear σ models with $(4, 0)$ or $(4, 4)$ supersymmetry [12,13], or when Tod [14] exhibited the relationship between a particular Einstein–Weyl geometry without torsion (the four-dimensional self-dual Einstein–Weyl geometry studied by Pedersen and Swann [15]) and local heterotic geometry (i.e. the Riemannian geometry with torsion and three complex structures, associated with $(4, 0)$ supersymmetric non-linear σ models [16–23]).

In [24,25], we analysed in a systematic way, first from a local point of view, then with completeness and compactness restrictions, the four-dimensional Einstein–Weyl structures equipped with a Bianchi metric. This allows us to illustrate the general results obtained by mathematicians around Gauduchon, Tod, Pedersen, Poon and Swann [15,26–32] and, e.g. to show that Einstein–Weyl structures equipped with a Bianchi metric are either conformally Einstein or conformally Kähler [24]. The aim of the present work is twofold.

- Extend our four-dimensional study to n dimensions, still in a local approach and, in the spirit of four-dimensional separation of "time" and "space", we restrict ourselves to cohomogeneity-one manifolds. In particular, we show that the main results proved by mathematicians for *compact Einstein–Weyl structures* hold true for (non-) compact cohomogeneity-one structures as soon as the $(n-1)$-dimensional principal orbit is a homogeneous reductive Riemannian space with a unimodular group of isometries.
  - ○ Existence of a Gauduchon [26] gauge such that the Weyl-form γ is co-closed, and such that the dual of the Weyl [28] form is a Killing vector for the metric.

- ◦ Still in that gauge, nice constraint on the conformal scalar curvature [27,29,30,33]:

$$S^D = -\frac{n(n-4)}{4}(\gamma_\nu \gamma^\nu) + \text{constant}.$$

- Get a better understanding of the symmetry that corresponds to the upper mentioned Killing vector.
  - ◦ One of our main results is a no-go theorem. If the $(n-1)$-dimensional Riemannian homogeneous space is the right coset space G/H, *a non-exact* ($\gamma \neq \mathrm{d}f$) *Einstein–Weyl space exists only if there exists* a non-empty subgroup H' of G such that $H \cap H' = \emptyset$, $[H, H'] = 0$.
  - ◦ We also prove that this isometry corresponds to the right-action of one of the generators of the subgroup H', and so that *the symmetry of the solution is bigger than that of the Einstein–Weyl equations*: it is enlarged from G acting on the left (G^{L}) to $G^{\mathrm{L}} \times GL(1, \mathbb{R})$. This unusual phenomenon, a kind of spontaneous generation of symmetry, results from the Einstein–Weyl constraints: note that such a phenomena will be helpful in the quantisation of the theory, as a Ward identity is more manageable than a geometrical constraint such as the Einstein–Weyl property.

The paper is organised as follows: in Section 2, we first recall the geometrical setting of Einstein–Weyl geometry and cohomogeneity-one metrics; then we emphasise some properties of left and right group action on coset spaces and finally we give the expressions of geometrical quantities, separating the n-dimensional metric g into a "time part" and a $(n-1)$-dimensional "space part". In Section 3, focussing on unimodular groups G, we exhibit a specific Gauduchon gauge and express the Einstein–Weyl equations in that gauge. In full generality, we are then able to prove the announced results (Lemma 1 and Theorem 1). We end the section by a characterisation of some special families of solutions where only two functions are involved in the expression of the Einstein–Weyl structure. In particular, we prove that for the whole family built on an $(n-2)$-dimensional compact symmetric Kähler space, the corresponding n-dimensional metric is locally conformally Kähler (Theorem 2).

Section 4 is then devoted to the family of $SU(m)$ left-invariant structures in $n = 2m$ dimensions. Thanks to the results of the previous section, the isometry group is enlarged to $U(1) \times SU(m)^{\mathrm{L}}$. As in the four-dimensional case, they are conformally Kähler and we obtain the explicit expression of the structure: it depends on three arbitrary parameters, up to a homothety. As in [25], we use the terminology of Gibbons and Hawking [34,35] on nuts and bolts to search for n-dimensional regular and complete solutions and show that, up to an arbitrary homothetic factor Γ, $(m+2)$ one-parameter families of solutions exist. In particular, we prove that a bolt(p)–bolt(p) solution exists iff the twist p is $1, 2, \ldots, (m-1)$. This proves one of Madsen's [31] conjectures.

The same work is done in Section 5 for $S^1 \times SO(n-1)$ left-invariant structures. We obtain the explicit expression of the structure depending on three arbitrary parameters, up to a homothety. Here again, we look for n-dimensional regular and complete solutions and show that, up to an arbitrary homothetic factor Γ, only three one-parameter families of non-conformally Einstein solutions exist, all with an everywhere positive conformal scalar curvature.

Some concluding remarks are offered in Section 6. Appendix A describes the splitting of n-dimensional geometric quantities to $(n-1)$-dimensional ones for cohomogeneity-one metrics and, in Appendix B, we relate two of our families of solutions of opposite orientations.

2. Einstein–Weyl structures and cohomogeneity-one metrics: the geometrical setting

2.1. Weyl space

A Weyl space [11,15] is a conformal manifold with a torsion-free connection D and a one-form γ such that for each representative metric g in a conformal class $[g]$,

$$D_\mu g_{\nu\rho} = \gamma_\mu g_{\nu\rho}. \tag{1}$$

A different choice of representative metric $g \to \tilde{g} = e^f g$ is accompanied by a change in $\gamma : \gamma \to \tilde{\gamma} = \gamma + \mathrm{d}f$. Conversely, if the one-form γ is exact, the metric g is conformally equivalent to a Riemannian metric $\tilde{g}$: $D_\mu \tilde{g}_{\nu\rho} = 0$. In that case, we shall speak of an exact Weyl structure.

The Ricci tensor associated to the Weyl connection D is defined by

$$[D_\mu, D_\nu]v^\rho = \mathcal{R}^{(D)\rho}_{\lambda,\mu\nu} v^\lambda, \quad \mathcal{R}^{(D)}_{\mu\nu} = \mathcal{R}^{(D)\rho}_{\mu,\rho\nu}. \tag{2}$$

$\mathcal{R}^{(D)}_{\mu\nu}$ is related to $R^{(\nabla)}_{\mu\nu}$, the Ricci tensor associated to the Levi-Civita connection:

$$\begin{aligned}\mathcal{R}^{(D)}_{\mu\nu} = R^{(\nabla)}_{\mu\nu} &+ \tfrac{1}{2}(n-1)\nabla_\nu\gamma_\mu - \tfrac{1}{2}\nabla_\mu\gamma_\nu + \tfrac{1}{4}(n-2)\gamma_\mu\gamma_\nu \\ &+ \tfrac{1}{2} g_{\mu\nu}[\nabla_\rho\gamma^\rho - \tfrac{1}{2}(n-2)\gamma_\rho\gamma^\rho].\end{aligned} \tag{3}$$

Using (2) and (3), a nice relation [33] constrains the conformally invariant two-form $\mathrm{d}\gamma$ which we call the field strength [1]

$$\begin{aligned}&\mathrm{d}\gamma = \tfrac{1}{2} F_{\mu\nu}\, \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu, \\ &g^{\mu\lambda} g^{\nu\rho} D_\lambda D_\rho F_{\mu\nu} = -\tfrac{1}{4}(n-4) F^{\mu\nu} F_{\mu\nu} \Leftrightarrow D_\mu D_\nu F^{\mu\nu} = -\tfrac{1}{4} n F^{\mu\nu} F_{\mu\nu}.\end{aligned} \tag{4}$$

2.2. The Gauduchon gauge

In the compact case, up to a homothety there exists a unique metric g in the conformal class such that γ is co-closed:

$$\nabla_\lambda \gamma^\lambda = 0.$$

(The Lorentz gauge for electromagnetism.)

[1] In the original point of view of Weyl, γ_μ is the electromagnetic field and in [33] the term Faraday's two-form is used for $\mathrm{d}\gamma$.

2.3. Einstein–Weyl spaces

Einstein–Weyl spaces are Weyl structures defined by [2]

$$\mathcal{R}^{(D)}_{\mu\nu} = \frac{S^D}{n} g_{\mu\nu} \Leftrightarrow R^{(\nabla)}_{\mu\nu} + \frac{n-2}{2}\left[\nabla_{(\mu}\gamma_{\nu)} + \frac{1}{2}\gamma_\mu\gamma_\nu\right] = \Lambda g_{\mu\nu},$$
$$\Lambda = \frac{S^D}{n} - \frac{1}{2}\left[\nabla_\lambda\gamma^\lambda - \frac{n-2}{2}\gamma_\lambda\gamma^\lambda\right]. \qquad (5)$$

Note that for an exact Einstein–Weyl structure, $\gamma = \mathrm{d}f$, the representative metric is conformally Einstein. Note also that the conformal scalar curvature is related to the scalar curvature through

$$S^D = g^{\mu\nu}\mathcal{R}^{(D)}_{\mu\nu} = n\Lambda + \tfrac{1}{2}n[\nabla_\lambda\gamma^\lambda - \tfrac{1}{2}(n-2)\gamma_\lambda\gamma^\lambda]$$
$$= R^{(\nabla)} + (n-1)[\nabla_\lambda\gamma^\lambda - \tfrac{1}{4}(n-2)\gamma_\lambda\gamma^\lambda]. \qquad (6)$$

For any Einstein–Weyl structure, another nice relation may be derived using the Bianchi identity

$$-\nabla_\nu\left[\frac{S^D}{n} + \frac{n-4}{4}\gamma_\lambda\gamma^\lambda\right] + \left(\nabla_\nu - \frac{1}{2}\gamma_\nu\right)(\nabla_\lambda\gamma^\lambda) = (\nabla^\lambda + \gamma^\lambda)(\nabla_{(\lambda}\gamma_{\nu)}). \qquad (7)$$

Notice that in a Gauduchon gauge and when the manifold is compact, [3] contraction of (7) with γ^ν, followed by an integration on the manifold, ensures that the vector γ_λ, dual of the Weyl-form γ, is a Killing vector [28].

A related relation is [27,29,33]

$$(\nabla_\nu + \gamma_\nu)\frac{S^D}{n} + \frac{1}{2}g^{\lambda\mu}D_\lambda F_{\mu\nu} = 0. \qquad (8)$$

2.4. Cohomogeneity-one metrics

Cohomogeneity-one metrics are real n-dimensional metrics with an isometry group G whose generic orbits are $(n-1)$ surfaces (we also restrict ourselves to effectively acting isometries, i.e. the isotropy subgroup H contains no non-trivial normal subgroups, discrete or not, of G [36]). This generalises to n-dimensions the homogeneity property of three-dimensional ordinary space in gravity and the n-dimensional distance is then split as

$$\mathrm{d}s^2 = g_{\mu\nu}\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu = (\mathrm{d}T)^2 + (\mathrm{d}\tau)^2 = (\mathrm{d}T)^2 + g_{\alpha\beta}\,\mathrm{d}x^\alpha\,\mathrm{d}x^\beta,$$
$$\mu,\nu = (0,\alpha), (0,\beta), \qquad (9)$$

where, given some "proper time" T, the T-fixed $(n-1)$ space will be a homogeneous space, i.e. a coset space G/H with G a connected group and H a closed subgroup. As

[2] $[a,b]$ and (a,b), respectively, mean antisymmetrisation and symmetrisation with respect to the indices a, b: $v_{(a}w_{b)} = \frac{1}{2}[v_a w_b + v_b w_a]$, etc.

[3] At least, as soon as integration by parts on the manifold is possible.

we consider only Riemannian manifolds, the isotropy subgroup H, being a subgroup of some orthogonal group, is compact [37]. The compactness of H ensures that G/H is a reductive homogeneous space [38], i.e. $\mathcal{G}$ and $\mathcal{H}$ denoting the corresponding Lie algebras, an invariant, non-degenerate, bilinear quadratic form on $\mathcal{G}$ exists and $\mathcal{G}$ may be decomposed according to

$$\mathcal{G} = \mathcal{H} \oplus \mathcal{M},$$

where $\mathcal{M}$ is $\mathrm{Ad}(H)$ invariant. So the commutation relations write

$$\begin{aligned}
&[h_a, h_b] = f^c_{ab} h_c, \quad h_a \in \mathcal{H}, \ \ a, b, c = 1, 2, \ldots, L, \ \ \dim H = L, \\
&[h_a, W_i] = f^j_{ai} W_j, \quad W_i \in \mathcal{M}, \ \ i, j, k = 1, 2, \ldots, (n-1), \\
&[W_i, W_j] = f^k_{ij} W_j + f^c_{ij} h_c.
\end{aligned} \tag{10}$$

A parameterisation of the $(n-1)$-homogeneous space is conveniently done through (right) equivalence classes in G/H in one to one correspondence with the considered point x on the $(n-1)$ surface

$$[L(x)] \in G, \quad L(x) \sim L'(x) \Leftrightarrow \exists h \in H, \ L(x) = L'(x) \cdot h.$$

The left-action of an arbitrary g_0 writes

$$[L(x')] = [g_0 \cdot L(x)] \Leftrightarrow L(x') = g_0 \cdot L(x) \cdot h^{-1}[x, g_0],$$

note that a left h_0 transformation, given by $L(x') = h_0 \cdot L(x) \cdot h_0^{-1}$, acts linearly on x.

The Lie algebra valued Maurer–Cartan one-form

$$M = L^{-1}(x)\, \mathrm{d}L(x)$$

defines the one-forms $e^i(x)$ and $\omega^a(x)$ by

$$M = e^i(x) W_i + \omega^a(x) h_a, \tag{11}$$

in particular, the one-forms $e^i(x) = e^i_\alpha(x)\, \mathrm{d}x^\alpha$ transform, under an arbitrary transformation of G, in an "homogeneous" way, according to

$$e^i(x) W_i \to e^i(x') W_i = h[x, g_0] \cdot e^i(x) W_i \cdot h^{-1}[x, g_0].$$

The infinitesimal version writes

$$\begin{aligned}
&\delta_{g_0} e^i(x) = -\epsilon^a(x, g_0) f^i_{aj} e^j, \\
&\quad \text{with} \quad h[x, g_0] = \exp[-\epsilon^a(x, g_0) h_a] \quad \text{and} \quad \epsilon^a(x, h_0) \equiv \epsilon^a(h_0).
\end{aligned} \tag{12}$$

Then, the most general G-invariant distance on the $(n-1)$-dimensional space may be written as

$$(\mathrm{d}\tau)^2 = h_{ij} e^i(x) e^j(x) \equiv h_{ij} e^i_\alpha(x) e^j_\beta(x)\, \mathrm{d}x^\alpha\, \mathrm{d}x^\beta, \tag{13}$$

where the symmetric, positive definite $(n-1)\times(n-1)$ tensor h_{ij} has to be invariant under H, i.e.

$$f_{ai}^{k}h_{kj}+f_{aj}^{k}h_{ik}=0, \tag{14}$$

and the $e_{\alpha}^{i}(x)$ are some vielbeins. Let η_{ij} be independent solutions of (14): they correspond to irreducible orthogonal representations (irreps) of the compact group H and may be used to write the H-invariant 2-tensor h_{ij} in block-diagonal form

$$\mathrm{d}\tau^{2}=\sum_{\eta=\text{irreps of }H}h^{\eta}\eta_{ij}e^{i}(x)e^{j}(x), \tag{15}$$

where η_{ij} is a positive definite symmetric matrix in the irreducible component labelled by η, and the h^{η}'s are some arbitrary positive parameters. The cohomogeneity-one requirement means that at any "proper time" T, the $(n-1)$-dimensional distance takes the form (13) and (15):

$$\begin{aligned}(\mathrm{d}s)^{2}&=(\mathrm{d}T)^{2}+(\mathrm{d}\tau)^{2}=(\mathrm{d}T)^{2}+h_{ij}[T]e^{i}(x)e^{j}(x)\\&=(\mathrm{d}T)^{2}+\sum_{\eta=\text{irreps of }H}h^{\eta}[T]\eta_{ij}e^{i}(x)e^{j}(x).\end{aligned}$$

Notice that there is no loss of generality in choosing the metric element $g_{00}=1$ as this corresponds to a choice of "proper time" T. The general analysis of Einstein–Weyl equations will use G-invariant cohomogeneity-one Weyl structure written as

$$\mathrm{d}s^{2}=\mathrm{d}T^{2}+h_{ij}(T)e^{i}e^{j},\qquad \gamma=\gamma_{0}(T)\,\mathrm{d}T+\gamma_{i}(T)e^{i} \tag{16}$$

with (h^{ij} is the matrix inverse of h_{ij})

$$f_{ai}^{k}h_{kj}[T]+f_{aj}^{k}h_{ik}[T]=0, \tag{17a}$$

$$f_{ai}^{j}\gamma_{j}(T)=0\overset{(17\mathrm{a})}{\Leftrightarrow}f_{ai}^{j}\gamma^{i}(T)=0,\quad \gamma^{i}=h^{ij}\gamma_{j}. \tag{17b}$$

Some inverse vielbeins E_{i}^{α} may be defined by

$$\mathrm{d}x^{\alpha}=E_{i}^{\alpha}e^{i}\Rightarrow e_{\alpha}^{i}E_{j}^{\alpha}=\delta_{j}^{i},\qquad e_{\alpha}^{i}E_{i}^{\beta}=\delta_{\alpha}^{\beta}. \tag{18}$$

The Maurer–Cartan consistency condition $\mathrm{d}M+M\wedge M=0$ gives

$$\begin{aligned}&\mathrm{d}e^{i}+\tfrac{1}{2}f_{jk}^{i}e^{j}\wedge e^{k}+f_{ak}^{i}\omega^{a}\wedge e^{k}=0\Rightarrow\nabla_{[\beta}e_{\alpha]}^{i}=\tfrac{1}{2}f_{jk}^{i}e_{\alpha}^{j}e_{\beta}^{k}+f_{ak}^{i}\omega_{[\alpha}^{a}e_{\beta]}^{k},\\&\mathrm{d}\omega^{a}+\tfrac{1}{2}f_{jk}^{a}e^{j}\wedge e^{k}+\tfrac{1}{2}f_{bc}^{a}\omega^{b}\wedge\omega^{c}=0\Rightarrow\nabla_{[\beta}\omega_{\alpha]}^{a}=\tfrac{1}{2}f_{jk}^{a}e_{\alpha}^{j}e_{\beta}^{k}+\tfrac{1}{2}f_{bc}^{a}\omega_{\alpha}^{b}\omega_{\beta}^{c}.\end{aligned} \tag{19}$$

Moreover (see Appendix A), using Eqs. (14) and (19), one obtains for the symmetrised covariant derivative

$$\nabla_{(\beta}e_{\alpha)}^{i}=-h^{ij}f_{j(l}^{k}h_{m)k}e_{\alpha}^{l}e_{\beta}^{m}-f_{am}^{i}\omega_{(\alpha}^{a}e_{\beta)}^{m}. \tag{20}$$

2.5. The right-action of G

For further use, note that a right-action of G on right equivalent classes in G/H can be defined for those elements h' of G (with h' not in H) that commute with all elements of H.[4] They form a subgroup H' of G. The corresponding Lie algebra elements belong to $(\mathcal{G}-\mathcal{H})$ and generate a subalgebra $\mathcal{H}'$; the Lie algebra of G may be decomposed according to

$$\mathcal{G} = \mathcal{H} \oplus \mathcal{H}' \oplus \mathcal{M},$$

where $\mathcal{M}$ is the complement of $\mathcal{H} \oplus \mathcal{H}'$. The commutation relations are now

$$\begin{aligned}
&[h_a, h_b] = f_{ab}^c h_c, \quad h_a \in \mathcal{H},\ a, b, c = 1, 2, \ldots, L,\ \dim H = L, \\
&[h'_u, h'_v] = f_{uv}^w h'_w, \quad h'_u \in \mathcal{H}',\ u, v, w = 1, 2, \ldots, L',\ \dim H' = L', \\
&[h_a, h'_u] = 0, \qquad [h_a, \tilde{W}_i] = f_{ai}^j \tilde{W}_j, \quad \tilde{W}_i \in \mathcal{M},\ i, j, k = 1, 2, \ldots, (n-1-L'), \\
&[h'_u, \tilde{W}_i] = f_{ui}^j \tilde{W}_j, \qquad [\tilde{W}_i, \tilde{W}_j] = f_{ij}^k \tilde{W}_j + f_{ij}^c h_c + f_{ij}^u h'_u.
\end{aligned} \tag{21}$$

In that case, note that this right-action of h' on right equivalence classes is simply defined by

$$[\tilde{L}(x')] = [\tilde{L}(x) \cdot h'_0] = [\tilde{L}(x)] \cdot h'_0.$$

Moreover, the Maurer–Cartan one-form M should now be decomposed as

$$M = \tilde{e}^i(x)\tilde{W}_i + y^u(x)h'_u + \tilde{\omega}^a(x)h_a. \tag{22}$$

On one hand, note that the one-forms $y^u(x)$ are left-invariant and the G-invariant cohomogeneity-one Weyl structure may be written as[5]

$$\mathrm{d}s^2 = \mathrm{d}T^2 + \tilde{h}_{ij}(T)\tilde{e}^i\tilde{e}^j + \tilde{h}_{uv}(T)y^u y^v, \qquad \gamma = \gamma_0(T)\,\mathrm{d}T + \tilde{\gamma}_u(T)y^u \tag{23}$$

with

$$f_{ai}^k\tilde{h}_{kj}[T] + f_{aj}^k\tilde{h}_{ik}[T] = 0. \tag{24}$$

On the other hand, under a right-action, the one-forms $\tilde{e}^i(x)$ and $y^u(x)$ transform linearly, and the $\omega^a(x)$ are invariant

$$\delta_{h'}\tilde{e}^i(x) = -\epsilon^u f_{uj}^i \tilde{e}^j(x), \qquad \delta_{h'}y^v(x) = -\epsilon^u f_{uw}^v y^w(x),$$

then, the Weyl structure (23) and (24) remains of the same form (with a tensor $\tilde{h}_{ij}[T]$ changed

[4] Of course, a right-action of all elements of G that normalise H may always be defined. However, in the analysis of the isometries of Einstein–Weyl structures, only those elements that commute with H will play a role as the corresponding one-forms are left-invariant.

[5] Using irreducible representations of H (see Section 2.4), $\tilde{h}_{ij}$ could be written as in (15); moreover, $\tilde{h}_{uv}$ is an arbitrary positive definite symmetric matrix.

into another H-invariant one, thanks to Jacobi identities) according to

$$\tilde{h}'_{ij} = \tilde{h}_{ij} + \epsilon^u [f^k_{ui}\tilde{h}_{kj} + f^k_{uj}\tilde{h}_{ik}], \qquad \tilde{h}'_{uv} = \tilde{h}_{uv} + \epsilon^w [f^{u'}_{wu}\tilde{h}_{u'v} + f^{u'}_{wv}\tilde{h}_{uu'}],$$
$$\tilde{\gamma}'_u = \tilde{\gamma}_u + \epsilon^v f^w_{vu}\tilde{\gamma}_w. \qquad (25)$$

(A discussion of non-linear σ models built on homogeneous spaces with such a H' subgroup may be found in [39, Sections 3.1 and 3.2]).

2.6. The n-dimensional geometric quantities

The n-dimensional geometric quantities may now be expressed as functions of the $(n-1)$-dimensional ones (Appendix A).

First, thanks to previous results (19) and (20), $R^{(\nabla)}_{\mu\nu}$ may be expressed as (e.g. see [24,40])

$$R^{(\nabla)}_{00} = -\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}T}\left(\frac{h'}{h}\right) - \frac{1}{4}K^j_i K^i_j, \quad K^j_i = \frac{\mathrm{d}h_{ik}}{\mathrm{d}T}h^{kj},\ h = \det[h_{ij}],\ h' = \frac{\mathrm{d}h}{\mathrm{d}T},$$
$$R^{(\nabla)}_{0\alpha} = \frac{1}{2}e^k_\alpha [f^i_{kj} - \delta^i_k f^m_{jm}]K^j_i,$$
$$R^{(\nabla)}_{\alpha\beta} = \sigma^i_\alpha \sigma^j_\beta \left[R^{(n-1)}_{ij} - \frac{1}{2}\frac{\mathrm{d}K_{ij}}{\mathrm{d}T} + \frac{1}{2}K^k_i K_{kj} - \frac{h'}{4h}K_{ij}\right], \quad K_{ij} = K^k_i h_{kj} = \frac{\mathrm{d}h_{ij}}{\mathrm{d}T}, \text{etc.} \qquad (26)$$

where $R^{(n-1)}_{ij}$ is the $(n-1)$-dimensional Ricci tensor associated to the homogeneous space Levi-Civita connection, in the vielbein basis e^i, may be expressed as a function of the metric h_{ij} and of the structure constants of the group [40,41].

Second, the Bianchi identity splits [24]

$$f^i_{jk}R^{(n-1)j}_i + f^i_{ji}R^{(n-1)j}_k = 0, \quad k = 1, 2, \ldots, n-1 \quad [40, \text{Eqs. } (116, 25)], \qquad (27)$$

$$h^{ij}\frac{\mathrm{d}}{\mathrm{d}T}R^{(n-1)}_{ij} \equiv \frac{\mathrm{d}R^{(n-1)}}{\mathrm{d}T} + K^j_i R^{(n-1)i}_j = 2(\nabla_\alpha E^\alpha_i)R^i_0 \qquad (28)$$

with

$$\nabla_\alpha E^\alpha_i = f^i_{ji} + (\omega^a_\beta E^\beta_l) f^l_{ai}, \qquad R^i_0 = h^{ij}E^\alpha_j R^{(\nabla)}_{0\alpha}, \qquad R^{(n-1)} = R^{(n-1)}_{ij}h^{ij}.$$

We do not find the nice equation (28) in the standard textbooks on gravity (even for ordinary four-dimensional space–time with a three-dimensional group of isometries, i.e. no subgroup H, where $\nabla_\alpha E^\alpha_i$ simplifies to f^i_{ji}).

Third, one may also obtain using (17a), (17b), (19) and (20)

$$\nabla_{(0}\gamma_{0)} = \frac{\mathrm{d}\gamma_0}{\mathrm{d}T}, \qquad \nabla_{(0}\gamma_{\alpha)} = \frac{1}{2}e^i_\alpha h_{ij}\frac{\mathrm{d}\gamma^j}{\mathrm{d}T}, \quad \gamma^i = h^{ij}\gamma_j,$$
$$\nabla_{(\alpha}\gamma_{\beta)} = e^i_\alpha e^j_\beta\left[\frac{1}{2}\gamma_0 K_{ij} + h_{l(i}f^l_{j)k}\gamma^k\right], \qquad \nabla_\mu\gamma^\mu = \frac{\mathrm{d}\gamma_0}{\mathrm{d}T} + \frac{h'}{2h}\gamma_0 + \gamma^i f^j_{ji}. \qquad (29)$$

Note that one may always choose a representative in the conformal class $[g]$ such that $\gamma_0(T) \equiv 0$. Then, as soon as $f_{ij}^{j} = 0,\ i = 1, 2, \dots, (n-1)$,[6] this choice gives a special family of Gauduchon gauges [24], and, in the rest of this study, we suppose that this condition is fulfilled.

3. The Einstein–Weyl equations in the special gauge $\gamma_0 = 0$

3.1. General results

For cohomogeneity-one metrics, the Einstein–Weyl equations (5) may be split and written in the special gauge $\gamma_0 = 0$ (let us recall that we consider algebras $\mathcal{G}$ with $f_{ij}^{j} = 0$):

$$\Lambda = -\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}T}\left(\frac{h'}{h}\right) - \frac{1}{4}K_i^{j}K_j^{i}, \tag{30}$$

$$0 = \frac{1}{2}f_{kj}^{i}K_i^{j} + \frac{n-2}{4}h_{ki}\frac{\mathrm{d}\gamma^i}{\mathrm{d}T}, \tag{31}$$

$$h_{ij}\Lambda = R_{ij}^{(n-1)} - \frac{1}{2}\frac{\mathrm{d}K_{ij}}{\mathrm{d}T} + \frac{1}{2}K_i^{k}K_{kj} - \frac{h'}{4h}K_{ij} + \frac{n-2}{2}\gamma^k h_{l(i}f_{j)k}^{l} + \frac{n-2}{4}\gamma_i\gamma_j. \tag{32}$$

On the one hand, the use of relations (28), (30) and (31) in the equations obtained through contraction of (32) with h^{ij} and K^{ij}, gives

$$\frac{\mathrm{d}}{\mathrm{d}T}\left[S^D + \frac{n(n-4)}{4}\gamma_i\gamma^i\right] = -\frac{n}{2}\frac{\mathrm{d}\gamma^i}{\mathrm{d}T}\left[\nabla_\alpha E_i^\alpha - \frac{n-4}{2}\gamma_i\right]. \tag{33}$$

On the other hand, Eq. (7) splits into

$$\frac{\mathrm{d}}{\mathrm{d}T}\left[S^D + \frac{n(n-4)}{4}\gamma_i\gamma^i\right] = -\frac{n}{2}\frac{\mathrm{d}\gamma^i}{\mathrm{d}T}[\nabla_\alpha E_i^\alpha + \gamma_i], \tag{34}$$

$$\frac{\mathrm{d}}{\mathrm{d}T}\left[h_{ij}\frac{\mathrm{d}\gamma^j}{\mathrm{d}T}\right] = \gamma^j[f_{ji}^{k}\gamma_k + X_{ij} + \nabla_\alpha E_k^\alpha (D_j)_i^k], \tag{35}$$

where the traceless matrices D_i have for matrix elements $(D_i)_m^n \equiv f_{im}^n + f_{ir}^s h_{sm}h^{rn}$ and X_{ij} is a symmetric, non-negative matrix

$$X_{ij} = f_{im}^n[f_{jn}^m + f_{jr}^s h_{ns}h^{mr}] = 2f_{i(m}^n h_{s)n} f_{jr}^s h^{rm} = \tfrac{1}{2}(D_i)_m^n(D_j)_n^m.$$

Indeed, V being any eigenvector of the symmetric matrix X with eigenvalue λ,

$$\sum_j X_{ij}(V)_j = \lambda(V)_i,$$

[6] As $f_{ia}^a = 0$ and (thanks to the compactness of H) $f_{aj}^j + f_{ab}^b = 0$, our condition reduces itself to the unimodularity of the adjoint action of G. This also means that the measure is G-invariant.

one can choose a diagonal basis for the positive definite metric h_{ij} and compute

$$\lambda\sum_i (V)_i(V)_i = \sum_{i,j} X_{ij}(V)_i(V)_j = \frac{1}{2}\sum_{s,t} h^{ss}h_{tt}\left[\sum_i (V)_i(D_i)\right]_s^t\left[\sum_j (V)_j(D_j)\right]_s^t \geq 0.$$

As a consequence, the eigenvalues of the matrix X are non-negative.

Moreover, the existence of a zero eigenvalue requires an eigenvector V satisfying

$$\left[\sum_i (V)_i(D_i)\right]_s^t = 0 \Leftrightarrow \sum_i (V)_i f_{i(m}^s h_{n)s} = 0.$$

Eqs. (33) and (34) give (as soon as $n \geq 3$)

$$\gamma_i\frac{\mathrm{d}\gamma^i}{\mathrm{d}T} = 0. \tag{36}$$

Then, contraction of (35) with γ^i using (17a) and (17b) and $\nabla_\alpha E_i^\alpha = (\omega_\beta^a E_l^\beta) f_{ai}^l$ leads to

$$\frac{\mathrm{d}\gamma^i}{\mathrm{d}T}h_{ij}\frac{\mathrm{d}\gamma^j}{\mathrm{d}T} + \gamma^i X_{ij}\gamma^j = 0, \tag{37}$$

which enforces

$$\frac{\mathrm{d}\gamma^i}{\mathrm{d}T} = 0 \Leftrightarrow \gamma^i = \Gamma^i \text{ constants constrained by (17a) and (17b)} : \Gamma^i f_{ai}^j = 0. \tag{38}$$

As a consequence, the operator

$$Z_\Gamma = \Gamma^i W_i \tag{39}$$

commutes with all h_a.

Then, a non-exact[7] Einstein–Weyl structure, i.e. a solution with at least one non-vanishing H-invariant $(n-1)$ vector Γ^i, requires that the algebra $(\mathcal{G}-\mathcal{H})$ contain some h' element (see Section 2.5) and that there exist at least one zero eigenvalue for X:

$$\Gamma^i(D_i)_m^s h_{ns} = \Gamma^i f_{i(m}^s h_{n)s} = 0. \tag{40}$$

Then we have the following lemma.

Lemma 1. *Given a homogeneous $(n-1)$-dimensional space G/H, the Lie algebra of G satisfying $\sum_j f_{ij}^j = 0$, $i, j = 1, \ldots, n-1$, an n-dimensional non-exact Einstein–Weyl structure of cohomogeneity-one under the left-action of G may exist only if at least one generator in $\mathcal{G}-\mathcal{H}$ commutes with all the generators of $\mathcal{H}$.*

[7] An exact Einstein–Weyl structure with a non-vanishing γ also requires the existence of some h' element in the algebra $(\mathcal{G}-\mathcal{H})$, but in that work, we consider mainly non-exact Einstein–Weyl structures.

In particular, as the structure constants of a symmetric coset space satisfy $f_{ij}^{k} = 0$, one gets the following corollary.

Corollary 1. *Any n-dimensional Einstein–Weyl structure of cohomogeneity-one under the action of a group G and whose principal orbit G/H is a symmetric space without flat factors, can only be an exact Einstein–Weyl structure.*

As a particular case, note that $G = SO(n)$ being the maximal isometry group of an $(n-1)$-dimensional homogeneous space [42], in that situation G/H will be the sphere S^{n-1}; but, an $SO(n-1)$-invariant Weyl-form γ reduces to $\gamma_0\,\mathrm{d}T$. So, the sole Weyl structure is an exact one, in agreement with Corollary 1.

Thanks to the discussion in Section 2.5, a right-action of Z_Γ may then be defined. Under an infinitesimal right group transformation $Z^{\mathrm{R}} = \exp[\epsilon Z_\Gamma]$, any representative of a right equivalence class in G/H transforms according to

$$L(x') = L(x) \cdot Z^{\mathrm{R}} \cdot h^{-1}(x, \Gamma),$$

and, with $h(x, \Gamma) = \exp[-\epsilon^a(x, \Gamma)h_a]$, the one-forms $e^i(x)$ and $\omega^a(x)$, still defined through (11), transform according to

$$\delta_{Z^{\mathrm{R}}} e^i(x) = -[\epsilon^b(x, \Gamma) f_{bj}^i + \epsilon \Gamma^k f_{kj}^i] e^j(x),$$
$$\delta_{Z^{\mathrm{R}}} \omega^a(x) = -[\epsilon^b(x, \Gamma) f_{bc}^a \omega^c(x) + \epsilon \Gamma^k f_{kj}^a e^j(x)] + \mathrm{d}\epsilon^a(x, \Gamma). \tag{41}$$

The "gauge" function $\epsilon^a(x, \Gamma)$ can be expressed as

$$\epsilon^a(x, \Gamma) = \epsilon \Gamma^i \omega_\alpha^a(x) E_i^\alpha(x). \tag{42}$$

Indeed,

$$L^{-1}(x) \cdot [L(x + \delta x) - L(x)] = Z^{\mathrm{R}} \cdot \exp[\epsilon^a(x, \Gamma)h_a] - 1 \simeq \epsilon \Gamma^k W_k + \epsilon^a(x, \Gamma)h_a,$$

and, when one uses the Maurer–Cartan one-form $M(x)$, the left-hand side expression writes

$$L^{-1}(x) \cdot [L(x + \delta x) - L(x)] \simeq [e_\alpha^i(x) W_i + \omega_\alpha^a(x) h_a]\delta x^\alpha;$$

identification allows the elimination of δx^α and gives the announced result (42).

Using (17a), (17b) and (40), the distance and Weyl-form are readily checked to be invariant, which shows that the symmetry group of the Einstein–Weyl structure is enlarged from G^{L} to $G^{\mathrm{L}} \times GL(1, \mathbb{R})$, as there exists a combination of the left- and right-action of Z_Γ which acts linearly [39].

Let us now make contact with the general results obtained for a compact n-dimensional Einstein–Weyl structure in the (unique) Gauduchon gauge. First, Eq. (34) gives, in agreement with [27,30,33]

$$S^D + \tfrac{1}{4} n(n-4) \Gamma^i h_{ij}(T) \Gamma^i = \text{constant}, \tag{43}$$

second, relations (29) and (40) lead to $\nabla_{(\mu}\gamma_{\nu)} = 0$, and enforce γ^μ to be a Killing vector for the metric, in agreement with [28], the corresponding isometry generator being

$$\tilde{Z}_\Gamma = \Gamma^i E_i^\alpha \frac{\partial}{\partial x^\alpha}. \tag{44}$$

Note that $\nabla_{(\alpha}\gamma_{\beta)} = 0$ also enforces γ^α to be a Killing vector on T-fixed surfaces. Thanks to Eq. (19), $\tilde{Z}_\Gamma$ acts on one-forms $e^i(x)$ and $\omega^a(x)$ according to

$$\begin{aligned}\tilde{Z}_\Gamma e^i &= -[\Gamma^m f^i_{mj} + (\Gamma^m E^\alpha_m \omega^b_\alpha) f^i_{bj}]e^j,\\ \tilde{Z}_\Gamma \omega^a &= -[(\Gamma^m E^\alpha_m \omega^b_\alpha) f^a_{bc}\omega^c + \Gamma^m f^a_{kj} e^j] + \mathrm{d}(\Gamma^m E^\alpha_m \omega^a_\alpha),\end{aligned} \tag{45}$$

and leaves the Weyl-form invariant as

$$\tilde{Z}_\Gamma[\Gamma^j h_{ij} e^i] = \Gamma^j \Gamma^m [h_{ij} f^i_{mn} + E^\alpha_m \omega^a_\alpha h_{ij} f^i_{an}]e^n = -\Gamma^j \Gamma^m h_{in} f^i_{mj} e^n = 0.$$

The identification $\exp[\epsilon \tilde{Z}_\Gamma] = Z^{\mathrm{R}}$ immediately results when one compares Eqs. (41), (42) and (45).

So, we have, with the notations of Eqs. (10), (11) and (16), the following theorem.

Theorem 1. *Given a reductive homogeneous $(n-1)$-dimensional space G/H, where H is a closed subgroup of the connected group G and G is not necessarily compact but its regular representation is supposed to be unimodular.*

1. *An n-dimensional non-exact Einstein–Weyl structure of cohomogeneity-one under the left-action of G may exist only if some generators of $(\mathcal{G} - \mathcal{H})$ commute with all the generators of $\mathcal{H}$ (let $\mathcal{H}'$ be the subalgebra of such generators, and L' its dimension).*
2. *h'_0, one of the generators of $\mathcal{H}'$, being chosen, in the particular Gauduchon gauge obtained for $\gamma_0 = 0$ the isometry group contains an extra $GL(1,\mathbb{R})$, corresponding to a right-action of h'_0.*
3. *In that gauge, the Weyl-form is dual to the Killing vector of the chosen h'_0; it is then given by $\gamma = \Gamma^i_0 h^0_{ij}(T)e^j$, where Γ^i_0 are constant parameters constrained by $\Gamma^i_0 f^j_{ai} = 0$, and the distance is written as*

$$(\mathrm{d}s)^2 = (\mathrm{d}T)^2 + h^0_{ij}[T]e^i e^j.$$

4. *The $GL(1,\mathbb{R}) \times H$-invariant metric $h^0_{ij}[T]$ is constrained by $f^k_{a(i}h_{j)k} = \Gamma^i_0 f^k_{i(m}h_{n)k} = 0$ and by the equations*

$$\begin{aligned}\Lambda' &= -\frac{1}{2}\Gamma^i\Gamma^j h_{ij} - \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}T}\left(\frac{h'}{h}\right) - \frac{1}{4}K^j_i K^i_j = \text{constant}, \qquad f^k_{ij}K^j_k = 0,\\ R^{(n-1)}_{ij} &= \Lambda' h_{ij} + \frac{1}{2}\frac{\mathrm{d}K_{ij}}{\mathrm{d}T} - \frac{1}{2}K^k_i K_{kj} + \frac{h'}{4h}K_{ij}\\ &\quad + \frac{1}{2}\Gamma^m_0\Gamma^n_0\left[h_{mn}h_{ij} - \frac{n-2}{2}h_{mi}h_{nj}\right].\end{aligned} \tag{46}$$

5. *Still in that gauge, the conformal scalar curvature satisfies*

$$S^D + \tfrac{1}{4}n(n-4)\Gamma_0^i h_{ij}^0(T)\Gamma_0^j = n\Lambda'. \tag{47}$$

6. *As explained in Section* 2.5, *the* $(L'-1)$ *extra generators of the subgroup* H' *offer right transformations from one solution with some* $\{h_{ij}[T], \gamma_i[T]\}$ *to another solution: although not conformally equivalent, these solutions are related and should be considered as physically equivalent*.

Let us now use the notations of Eqs. (15) and (21)–(25) and select h'_{u_0}, one of the generators of $\mathcal{H}'$; let y^{u_0} be the corresponding one-form vielbein defined through the Maurer–Cartan one-form M (22). The $G^{\mathrm{L}} \times GL(1,\mathbb{R})$-invariant Einstein–Weyl structure may be rewritten — using irreducible representations of $H \times GL(1,\mathbb{R})$ — in a block-diagonal form

$$(\mathrm{d}s)^2 = (\mathrm{d}T)^2 + \tilde{h}_{ij}^{u_0}[T]\tilde{e}^i\tilde{e}^j + \tilde{h}_0[T]y^{u_0}y^{u_0} + \tilde{h}_{uv}[T]y^u y^v,$$
$$\gamma = \Gamma_{u_0}\tilde{h}_0[T]y^{u_0}, \quad u, v = 1, \dots, L'-1,\ i, j = 1, \dots, n-1-L', \tag{48}$$

where

- Γ_{u_0} is an arbitrary real parameter;
- $\tilde{h}_0[T]$ is an arbitrary positive function;
- $\tilde{h}_{ij}^{u_0}$ is a symmetric $(n-1-L') \times (n-1-L')$ 2-tensor, invariant under $\tilde{H} = H \times GL(1,\mathbb{R})$,

$$\tilde{h}_{ij}^{u_0}[T]e^i(x)e^j(x) = \sum_{\eta=\text{irreps of } \tilde{H}} \tilde{h}^{\eta}[T]\eta_{ij}e^i(x)e^j(x),$$

- $\tilde{h}_{uv}$ is a symmetric $(L'-1) \times (L'-1)$ 2-tensor, invariant under $GL(1,\mathbb{R})$,

$$\tilde{h}_{uv}[T]y^u(x)y^v(x) = \sum_{\rho=\text{irreps of } GL(1,\mathbb{R})} \tilde{h}^{\rho}[T]\rho_{uv}y^u(x)y^v(x),$$

- of course, the Einstein–Weyl equations (46) should also be imposed;
- y^{u_0} satisfies the Maurer–Cartan consistency condition

$$\mathrm{d}y^{u_0} = -\tfrac{1}{2}f_{vw}^{u_0}y^v \wedge y^w - \tfrac{1}{2}f_{ij}^{u_0}\tilde{e}^i \wedge \tilde{e}^j. \tag{49}$$

3.2. Some families of solutions

To escape from the no-go theorem of Corollary 1, it may be tempting to consider a non-semi-simple group $G \equiv GL(1,\mathbb{R}) \times \tilde{G}$ where $(\tilde{G}/H)$ is an $(n-2)$-dimensional symmetric space: in that case (note that the unimodularity condition $f_{ij}^j = 0$ is trivially satisfied) there are only two unknown functions of T : $\tilde{h}_0[T]$ and the one that multiplies the unique standard metric on $(\tilde{G}/H)$. A particular situation in that family is one, with a compact group G, considered by Madsen et al. [31,32] and analysed in Section 5 of the present work: there $G \equiv S^1 \times SO(n-1)$, $H \equiv SO(n-2)$. A four-dimensional non-compact example is the Bianchi VIII case [24] where $(\tilde{G}/H) \equiv SU(1,1)/U(1)$.

Other situations with only two unknown functions of T in (48) occur when $L'=1$ and $G/(H\times U(1))$ is a compact irreducible symmetric space: this requires $\mathcal{H}'=\mathcal{U}(1)\simeq\mathcal{SO}(2)\simeq S^1$. Indeed, this ensures that the matrix $h_{ij}[T]$ depends on a single function of T.[8] In that case, the subgroup $\tilde{H}$ contains an $U(1)$ factor, and the symmetric space $G/\tilde{H}$ is necessarily [44,45] a Kähler space whose Kähler form J is proportional to the closed 2-form:

$$\mathrm{d}y^{u_0}\overset{(49)}{=}-\tfrac{1}{2}f_{ij}^{u_0}\tilde{e}^i\wedge\tilde{e}^j=2J. \tag{50}$$

Let us explicitly prove that the Einstein–Weyl metrics in that family are Riemannian conformally Kähler metrics, so generalising our four-dimensional analysis [24]. The metric (48) writes

$$(\mathrm{d}s)^2=(\mathrm{d}T)^2+\tilde{h}_0(T)y^{u_0}y^{u_0}+2\tilde{h}_1(T)g_{i\bar{j}}\,\mathrm{d}z^i\,\mathrm{d}\bar{z}^{\bar{j}},$$

and the Kähler form $J\equiv \mathrm{i}g_{i\bar{j}}\,\mathrm{d}z^i\wedge d\bar{z}^{\bar{j}}$. $K(z,\bar{z})$ being the Kähler potential, the one-form y^0 writes

$$y^0=\mathrm{d}U-\mathrm{i}\partial_i K\,\mathrm{d}z^i+\mathrm{i}\partial_{\bar{j}}K\,\mathrm{d}\bar{z}^{\bar{j}},$$

and in the basis $\mathrm{d}x^m:\{\mathrm{d}T,\mathrm{d}U,\mathrm{d}z^i,\mathrm{d}\bar{z}^{\bar{j}}\}$, $i,\bar{j}=1,2,\ldots,\frac{1}{2}(n-2)$, $m=1,2,\ldots,n$, the metric will be written $(\mathrm{d}s)^2=G_{mp}\,\mathrm{d}x^m\,\mathrm{d}x^p$. Consider now the 2-form

$$\Omega=\sqrt{\tilde{h}_0(T)}\,\mathrm{d}T\wedge y^0+\tilde{h}_1(T)J.$$

In the basis $\{\mathrm{d}x^m\}$, $\Omega=\frac{1}{2}\bar{J}_{mp}\,\mathrm{d}x^m\wedge\mathrm{d}x^p$ defines an antisymmetric 2-tensor $\bar{J}_{mp}$. The tensor $\bar{J}_m^p=\bar{J}_{mq}G^{qp}$ is found to be

$$\bar{J}_m^p=\begin{bmatrix} 0 & \dfrac{1}{\sqrt{\tilde{h}_0}} & 0 & 0\\ -\sqrt{\tilde{h}_0} & 0 & 0 & 0\\ \mathrm{i}\sqrt{\tilde{h}_0}\partial_i K & -\partial_i K & \mathrm{i}\mathbb{I} & 0\\ -\mathrm{i}\sqrt{\tilde{h}_0}\partial_{\bar{j}}K & -\partial_{\bar{j}}K & 0 & -\mathrm{i}\mathbb{I}, \end{bmatrix} \tag{51}$$

and one verifies that $\bar{J}_m^p\bar{J}_p^q=-\delta_m^q$. With expression (51) for $\bar{J}_m^p$, one computes the Nijenhuis tensor and finds it to be identically zero: we have a complex structure, and, thanks to the antisymmetry of $\bar{J}_{mp}$, the metric G_{mp} is Hermitian with respect to $\bar{J}$. The differential $\mathrm{d}\Omega$ is computed and found to be

$$\mathrm{d}\Omega=\frac{\mathrm{d}\Phi}{\mathrm{d}T}\mathrm{d}T\wedge\Omega\quad\text{with}\quad\frac{\mathrm{d}\Phi}{\mathrm{d}T}=\frac{d\log\tilde{h}_1(T)}{\mathrm{d}T}-2\frac{\sqrt{\tilde{h}_0(T)}}{\tilde{h}_1(T)},$$

[8] The unimodularity condition decomposes into $f_{ij}^j+f_{iu_0}^{u_0}=0$, which is true (the indices i,j,k run among $(\mathcal{G}-\mathcal{H}-\mathcal{H}')$ generators), and $f_{u_0 j}^j=0$ which results from the compactness of H'.

and, after the conformal transformation, compatible with the cohomogeneity-one property ($\bar{J}^p_m$ being unchanged)

$$g \to \tilde{g} = g \exp[-\Phi(T)], \qquad \Omega \to \tilde{\Omega} = \Omega \exp[-\Phi(T)],$$
$$\gamma \to \tilde{\gamma} = \gamma - \mathrm{d}\Phi(T),$$

one gets $\mathrm{d}\tilde{\Omega} = 0$. As a consequence, we have the following theorem.

Theorem 2. *Given an arbitrary* $(n-2)$*-dimensional compact symmetric Kähler space* $G/\tilde{H}$ [*then* $\tilde{H} \equiv U(1) \times H$], *any non-exact Einstein–Weyl structure of cohomogeneity-one under the left-action of G has a Riemannian conformally Kähler metric and the principal orbit is the coset space G/H.*

Some remarks are in order:

- Note that we only used the cohomogeneity-one structure and the existence of an extra Killing vector for Einstein–Weyl structures.
- The structures are not "locally conformal Kähler" ones in the sense of Vaisman [15,46] as the complex structure is not covariantly constant with respect to the Weyl covariant derivative $D^{\tilde{\gamma}}$ (this would require $\gamma = \mathrm{d}\Phi(T)$).
- A particular situation in that family is the one where, in even dimensions $n = 2m$, $G = SU(m)$, $H = SU(m-1)$, $H' = U(1)$: it was considered by Madsen et al. [31,32] and is analysed in Section 4 of the present work ($G/\tilde{H} \equiv \mathbb{C}P^{m-1}$).
- Another one would be $G = SO(m+1)$, $H = SO(m-1)$, $H' = SO(2)$ [15], etc.

In other situations, the condition $H' = GL(1,\mathbb{R})$, will be relaxed, e.g. in dimensions $n = 5 + 4p$, with $G/H \equiv SU(p+2)/SU(p)$, $H' = SU(2)$, etc.

We do not intend to give here a complete classification of (non-exact) Einstein–Weyl structures in an arbitrary dimension, but mainly to emphasise that the symmetry of Einstein–Weyl solutions is bigger than that of the equations.

4. *SU(m)*-invariant structures

In $n = 2m$ dimensions, the previous analysis shows that a non-exact Weyl structure of cohomogeneity-one under $SU(m)$ has, in a Gauduchon gauge, an extra $U(1)$ invariance, so extending previous results shown for $n = 4$ [47]. The Weyl structure (48) may be written as

$$\mathrm{d}s^2 = (\mathrm{d}T)^2 + f^2(T)(y^0)^2 + h^2(T)g_{\mathrm{B}}, \qquad \gamma = \pm\Gamma f^2(T)y^0, \tag{52}$$

where Γ is a constant positive parameter, g_{B} is the standard Fubini-Study metric on CP^{m-1} with Kähler form J, Ricci curvature $= 2mg_{\mathrm{B}}$ and the one-form y^0 is chosen to satisfy $\mathrm{d}y^0 = 2J_{\mathrm{B}}$ (50), i.e. $\eta = 1$ in the notations of [31]. (Note that Madsen's parameter $\eta^2 = k^2$ may be reabsorbed into the definition of σ: all his equations are invariant under the change $f^2 \to f^2/k^2$, $\beta \to \beta/k$ such that $f^2\sigma^2$ and $\beta\sigma$ are left unchanged.) Note that here $\mathrm{d}y^0 \neq 0$, an exact Einstein–Weyl structure requires $\Gamma = 0$.

4.1. Local expressions

The Einstein–Weyl equations (46) write [31,32]

$$\Lambda' = -\frac{f''}{f} - (n-2)\frac{h''}{h} - \frac{1}{2}\Gamma^2 f^2,$$

$$c_{(00)}: \quad \Lambda' = -\frac{f''}{f} - (n-2)\frac{h' f'}{hf} + (n-2)\frac{f^2}{h^4} + \frac{n-4}{4}\Gamma^2 f^2,$$

$$c_{(ij)}: \quad \lambda' = -\frac{h''}{h} - (n-3)\frac{h'^2}{h^2} - \frac{h' f'}{hf} - 2\frac{f^2}{h^4} + \frac{n}{h^2} - \frac{1}{2}\Gamma^2 f^2. \tag{53}$$

To follow as closely as possible our previous four-dimensional analysis, [9] we rewrite g_{B} as $\frac{1}{4}(\mathrm{d}\tau)^2$, J_{B} as $\frac{1}{4}J$, y^0 as $\frac{1}{2}\sigma^3$, $\mathrm{d}\sigma^3 = J$ and Eq. (52) with notations inspired by gravitation [48,49]:

$$\mathrm{d}s^2 = \left[\omega^2(t)\omega_3(t)(\mathrm{d}t)^2 + \frac{\omega^2(t)}{\omega_3(t)}(\sigma^3)^2\right] + \omega_3(t)(\mathrm{d}\tau)^2, \quad \gamma = \pm\Gamma\frac{\omega^2(t)}{\omega_3(t)}\sigma^3. \tag{54}$$

As in [24], define $u(t)$ through

$$u(t) = \frac{1}{\omega_3\omega^2}\left(\frac{\mathrm{d}\omega_3}{\mathrm{d}t} - \omega^2\right). \tag{55}$$

The difference of the first two equations (53), allows the calculation of the derivative of $u(t)$:

$$\frac{\mathrm{d}u}{\mathrm{d}t} = -\frac{1}{2}\omega^2[\Gamma^2 + u^2] < 0.$$

Then, one can change the variable t into u and compute

$$\frac{\mathrm{d}\omega_3}{\mathrm{d}u} = -2\frac{1 + u\omega_3}{\Gamma^2 + u^2},$$

which integrates to

$$\omega_3(u) = 2\frac{k - u}{\Gamma^2 + u^2}. \tag{56}$$

Defining

$$\Omega^2 = \tfrac{1}{4}(\Gamma^2 + u^2)\omega^2, \tag{57}$$

and using the Einstein–Weyl equations (53), one obtains a second-order linear differential

[9] There, $(\mathrm{d}\tau)^2 = \sigma_1^2 + \sigma_2^2$, where σ_i, $i = 1, 2, 3$ are three $SU(2)$ left-invariant one-forms satisfying $\mathrm{d}\sigma_i = \frac{1}{2}\epsilon_{ijk}\sigma_j \wedge \sigma_k$.

equation

$$\frac{\mathrm{d}^2\Omega^2}{\mathrm{d}u^2} - \left[\frac{2(m-3)u}{\Gamma^2+u^2} + \frac{m-4}{k-u}\right]\frac{\mathrm{d}\Omega^2}{\mathrm{d}u} - \left[\frac{3(m-2)(\Gamma^2+k^2)}{(k-u)^2} + m\right]\frac{\Omega^2}{\Gamma^2+u^2} = -\frac{m}{\Gamma^2+u^2}. \tag{58}$$

Excluding Einstein solutions (as well as exact Einstein–Weyl structures), we rescale u and k according to $u = \Gamma x,\ k = \Gamma\kappa$, and get the following (Γ-independent) expression for Ω^2:

$$\Omega^2(x) = m\left(\frac{1+x^2}{\kappa - x}\right)^{m-2}[l_1(x^2-2\kappa x-1) + I_m[\kappa,x] - 2l_2 I_{m+1}[\kappa,x]], \tag{59}$$

where $n \geq 2$:

$$I_n[\kappa,x] = \frac{(\kappa-x)^{n-2}}{2[1+x^2]^{n-2}} + (x^2-2\kappa x-1)\left[(n-2)\int_x^\kappa \frac{(\kappa-y)^{n-3}}{2[1+y^2]^{n-1}}\,\mathrm{d}y + \frac{\delta_{n,2}}{2(1+\kappa^2)}\right]. \tag{60}$$

For further use, notice that the functions $I_n[\kappa,x]$ may be expressed as

$$I_n[\kappa,x] = (x^2-2\kappa x-1)J_n(\kappa,x) \quad \text{with} \quad \left.\frac{\partial J_n(\kappa,x)}{\partial x}\right|_\kappa = \frac{(\kappa-x)^{n-1}}{(x^2-2\kappa x-1)^2[1+x^2]^{n-2}} > 0. \tag{61}$$

When $x \to -\infty$, the functions $J_n(\kappa,x)$ become

$$\tilde{J}_n(\kappa) = \delta_{n,2}\frac{1}{2(1+\kappa^2)} + (n-2)\int_{-\infty}^\kappa \frac{(\kappa-y)^{n-3}}{2[1+y^2]^{n-1}}\,\mathrm{d}y > 0, \tag{62}$$

and one proves that

$$J_n(\kappa,x) - \tilde{J}_n(\kappa) \simeq \frac{1}{n(-x)^n}, \quad x \to -\infty. \tag{63}$$

The behaviour near κ is

$$J_n(\kappa,x) \simeq -\frac{(\kappa-x)^n}{n(1+\kappa^2)^n}, \quad x \to \kappa^-. \tag{64}$$

Then $J_n(\kappa,x)$ is an increasing function from $\tilde{J}_n(\kappa)$ to $+\infty$ when x varies from $-\infty$ to $(\kappa - \sqrt{1+\kappa^2})$, and from $-\infty$ to zero when x varies from $(\kappa - \sqrt{1+\kappa^2})$ to κ. As a consequence, $I_n(\kappa,x)$ is a continuous positive function between $-\infty$ and κ where it vanishes. These properties will be useful in the discussion of the regularity of the distance.

Eqs. (56) and (59) and

$$\frac{\mathrm{d}u}{\mathrm{d}t} = -2\Omega^2 \tag{65}$$

give the distance [10] and Weyl-form as functions of the new "proper time" x:

$$\mathrm{d}s^2=\frac{2}{\Gamma}\left[\frac{\kappa-x}{\Omega^2(1+x^2)^2}(\mathrm{d}x)^2+\frac{\Omega^2}{\kappa-x}(\sigma^3)^2+\frac{\kappa-x}{(1+x^2)}(\mathrm{d}\tau)^2\right],$$
$$\gamma=\pm\frac{2\Omega^2}{\kappa-x}\sigma^3. \quad (66)$$

Finally, the conformal scalar curvature is computed from (47) and (53)

$$S^D=m^2l_2\Gamma-2m(m-2)\Gamma\frac{\Omega^2}{\kappa-x}=\frac{n\Gamma}{4}\left[nl_2-2(n-4)\frac{\Omega^2}{\kappa-x}\right]\le\frac{n^2\Gamma l_2}{4}. \quad (67)$$

If one looks for solutions with a constant conformal scalar curvature, Eq. (58) can only be satisfied for $m=2$ [33,43].

As discussed in Section 3.2 and Theorem 2, under the conformal transformation $\tilde{g}=\frac{1}{2}\Gamma[1+x^2]g$, the metric may be rewritten in the standard form (54) with

$$\tilde{\omega}=\sqrt{\Omega^2(1+x^2)},\qquad \tilde{\omega}_3=\kappa-x,$$

the "proper time" $\tilde{t}$ being given by

$$\mathrm{d}\tilde{t}=-\frac{\mathrm{d}x}{\Omega^2(1+x^2)}.$$

Then,

$$\frac{\mathrm{d}\tilde{\omega}_3}{\mathrm{d}\tilde{t}}-\tilde{\omega}^2=0,$$

ensuring that the n-dimensional metric $\tilde{g}$ is Kähler with Kähler form given by

$$\tilde{J}^n=\tilde{\omega}^2\,\mathrm{d}\tilde{t}\wedge\sigma^3+\tilde{\omega}_3J,\qquad \mathrm{d}\sigma^3=J.$$

Then we have proved the following theorem.

Theorem 3. *The most general 2m-dimensional* (*non-*) *compact non-exact Einstein–Weyl structure with isometry* $SU(m)$, $m\ge 2$, *is a 3-parameter structure* (*plus one homothetic parameter*): *the metric is locally conformally Kähler*.

The conformal scalar curvature is a constant in the Gauduchon gauge if and only if $n=4$ *dimensions*.

In the following section, we shall consider the possible positive definite and regular $U(m)$-invariant Einstein–Weyl metrics. In his Ph.D., Madsen gives a classification of compact solutions. Here, in the same spirit as in [25], we use the terminology of Gibbons and Hawking [34,35] on nuts and bolts, well adapted to the analysis of the completeness of our candidate metrics on orientable manifolds. We shall prove that, up to an arbitrary homothetic factor $\Gamma>0$, there exist $m+2$ one-parameter families of complete Einstein–Weyl metrics

[10] Of course, the parameters κ, l_1, l_2 and the proper time x are constrained by positivity $\Omega^2>0$, $\kappa-x>0$.

with a non-exact Weyl form, each depending on a strictly positive constant l_2 related to the conformal scalar curvature.

4.2. Regular metrics

The function $\Omega^2(x)$ has to be positive on the proper time interval, which is then limited by its zeroes, and let us recall that positivity also requires $x < \kappa$. So, only four kinds of proper time interval may occur: $]-\infty, \kappa[$, $[-\infty, x_0[$, $]x_0', \kappa[$ and $]x_0', x_0[$.

The possible singularities of the distance (66) occur at $-\infty$, κ or at a zero of the function $\Omega^2(x)$.

(a) *Regularity of the distance as* $x \to -\infty$. When $x \to -\infty$, $\Omega^2(x) \simeq m\delta(-x)^m$ where $\delta = l_1 + \tilde{J}_m(\kappa) - 2l_2\tilde{J}_{m+1}(\kappa)$. The behaviour of the distance is readily seen to be singular if $\delta \neq 0$. Indeed,

$$\mathrm{d}s^2 \sim \frac{2}{\Gamma}\left[\frac{(\mathrm{d}x)^2}{m\delta(-x)^{(m+3)}} + m\delta(-x)^{m-1}(\sigma^3)^2 + \frac{1}{(-x)}(\mathrm{d}\tau)^2\right], \tag{68}$$

and the change of variable $\rho = (-x)^{-(m+1)/2}$ leaves a non-removable singularity at $\rho = 0$.

Consider now the special case when δ vanishes: thanks to (63), the function $\Omega^2(x)$ goes to 1 when $x \to -\infty$. So, the distance behaves as

$$\mathrm{d}s^2 \sim \frac{2}{\Gamma}\left[\frac{(\mathrm{d}x)^2}{(-x)^3} + \frac{1}{(-x)}[(\sigma^3)^2 + (\mathrm{d}\tau)^2]\right], \tag{69}$$

and, after the change $\rho = (-x)^{-1/2}$:

$$\mathrm{d}s^2 \sim \frac{8}{\Gamma}\left[(\mathrm{d}\rho)^2 + \frac{\rho^2}{4}[(\sigma^3)^2 + (\mathrm{d}\tau)^2]\right], \quad \rho \to 0,$$

the singularity is removable if one chooses Cartesian co-ordinates rather than polar ones. Near the end point $\rho \to 0$, the manifold is a point which gives a nut [34,35]. To sum up, we have the following lemma.

Lemma 2. *If the proper time interval extends down to* $-\infty$, *the metric can be regular only if* $\delta \equiv l_1 + \tilde{J}_m(\kappa) - 2l_2\tilde{J}_{m+1}(\kappa) = 0$, *and then a nut occurs.*

(b) *Regularity of the distance at* $x = \kappa$. Consider now the behaviour of the distance near $x = \kappa$ supposed to be the highest possible value of the proper time compatible with a positive metric. As $\Omega^2(x) \simeq m(1+\kappa^2)^{m-1}(-l_1)(\kappa - x)^{-m+2}$, the behaviour of the distance is readily seen to be singular if $l_1 \neq 0$. Indeed,

$$\mathrm{d}s^2 \sim \frac{2}{\Gamma}\left[\frac{(\kappa - x)^{m-1}(\mathrm{d}x)^2}{-ml_1(1+\kappa^2)^{(m+1)}} - ml_1\left(\frac{1+\kappa^2}{\kappa - x}\right)^{m-1}(\sigma^3)^2 + \frac{\kappa - x}{1+\kappa^2}(\mathrm{d}\tau)^2\right], \tag{70}$$

and the change of variable $\rho = (\kappa - x)^{(m+1)/2}$ leaves a non-removable singularity at $\rho = 0$.

We are left with the case $l_1 = 0$, where, thanks to (64), one finds

$$\Omega^2(x) \simeq \frac{(\kappa - x)^2}{1+\kappa^2}.$$

One can change the variable x into ρ given by $\rho = \sqrt{\kappa - x}$ and, here also, get the following nut behaviour for the distance near $x = \kappa$:

$$\mathrm{d}s^2 \sim \frac{8}{\Gamma[1+\kappa^2]}\left[(\mathrm{d}\rho)^2 + \frac{\rho^2}{4}[(\sigma^3)^2 + (\mathrm{d}\tau)^2]\right], \quad \rho \to 0.$$

To sum up, we have the following lemma.

Lemma 3. *If the proper time interval extends up to κ, the metric can be regular only if $l_1 = 0$, and then a nut occurs.*

(c) *Regularity of the distance at a zero of $\Omega^2(x)$.* At last, singularities in the distance may occur at zeroes of $\Omega^2(x)$. If $\Omega^2(x_0) = 0$ with $(\mathrm{d}\Omega^2/\mathrm{d}x)(x_0) = 0$ and $x_0 \neq \kappa$, the differential equation (58) enforces x_0 to be a maximum, which contradicts positivity.

So, consider the situation with $(\mathrm{d}\Omega^2/\mathrm{d}x)(x_0) \neq 0$ and change the variable x to ρ according to

$$x = x_0 + \rho^2 \frac{\mathrm{d}\Omega^2}{\mathrm{d}x}(x_0), \tag{71}$$

using $\Omega^2(x) \simeq \rho^2[(\mathrm{d}\Omega^2/\mathrm{d}x)(x_0)]^2$, the distance behaves when $\rho \to 0$ as

$$\mathrm{d}s^2 \sim \frac{8(\kappa - x_0)}{\Gamma[1+x_0^2]^2}\left[(\mathrm{d}\rho)^2 + \rho^2\left(\left(\frac{1+x_0^2}{\kappa - x_0}\right)\frac{\mathrm{d}\Omega^2}{\mathrm{d}x}(x_0)\right)^2\left(\frac{\sigma^3}{2}\right)^2 + \frac{1+x_0^2}{4}(\mathrm{d}\tau)^2\right], \tag{72}$$

and one has the following lemma.

Lemma 4. *If the function $\Omega^2(x)$ vanishes at x_0, the metric can be regular only if x_0 is a bolt of twist p, i.e.*

$$\Omega^2(x_0) = 0, \qquad \left(\frac{1+x_0^2}{\kappa - x_0}\right)\frac{\mathrm{d}\Omega^2}{\mathrm{d}x}(x_0) = \pm p, \quad p = 1, 2, \ldots. \tag{73}$$

Indeed, in such a case, restricting the range in the angle $\psi \in [0, 4\pi]$ involved in $\sigma^3 = \mathrm{d}\psi + \cos\theta\,\mathrm{d}\phi$, etc. to the interval $[0, 4\pi/p]$, and changing from polar co-ordinates $(\rho, \psi/2)$ to Cartesian ones, there is no longer a singularity in the distance when ρ goes to zero. Near the end point $\rho \to 0$, the manifold is $\mathbb{C}P^{m-1}$. Moreover, the $U(1)$ isometry corresponding to changes in ψ becomes $U(1)/\mathbb{Z}_p$.

If $(\mathrm{d}\Omega^2/\mathrm{d}x)(x_0) > 0$ the bolt is $+p$ and the proper time interval extends to κ or to another bolt$-p$ at $x_1 > x_0$; on the other hand, the bolt at x_0 is a $-p$ one and the proper time interval extends down to $-\infty$ or to another bolt$(+p)$ at $x_2 < x_0$.

Condition (73) may be rewritten as a relation between κ, l_2 and x_0 as

$$\kappa - x_0 = (m \pm p)\frac{1+x_0^2}{2(\mp p x_0 + m l_2)}. \tag{74}$$

We have now all the building blocks needed in our discussion on the regularity of our Einstein–Weyl metrics (note that the positive function $\|\gamma\|^2 = 2\Gamma\Omega^2(x)/(\kappa - x)$ vanishes at both ends of the allowed proper time intervals).

4.2.1. Nut–nut metric

Consider firstly the situation where the allowed range of x be the largest one, $]-\infty, \kappa]$. According to Lemma 3, $l_1 = 0$ for completeness at $x = \kappa$; moreover, a nut at $-\infty$ requires (Lemma 2)

$$l_1 + \tilde{J}_m(\kappa) - 2l_2\tilde{J}_{m+1}(\kappa) = 0.$$

Hopefully, for a given value of the parameter l_2, the vanishing of $f_m(\kappa) \equiv \tilde{J}_m(\kappa) - 2l_2\tilde{J}_{m+1}(\kappa)$, will determine a unique value for the parameter κ.

Notice firstly that, thanks to the positivity of the $\tilde{J}_n$, the function $f_m(\kappa)$ can vanish only when $l_2 > 0$. After an integration by parts, $f_m(\kappa)$ may be rewritten as

$$f_m(\kappa) = -(m-1)\int_{-\infty}^{\kappa} \frac{(y+l_2)[\kappa - y]^{m-2}}{[1+y^2]^m}\,\mathrm{d}y. \tag{75}$$

On one hand, this function cannot vanish if $\kappa + l_2 \leq 0$. On the other hand, after some algebra, one obtains the differential equation

$$(m-2)f_m(\kappa) - (\kappa + l_2)\frac{\mathrm{d}f_m(\kappa)}{\mathrm{d}\kappa} = A > 0,$$
$$A = \delta_{m,2}\frac{(\kappa+l_2)^2}{(1+\kappa^2)^2} + (m-1)(m-2)\int_{-\infty}^{\kappa} \frac{(y+l_2)^2[\kappa-y]^{m-3}}{[1+y^2]^m}\,\mathrm{d}y. \tag{76}$$

Then, as the vanishing of f_m requires $(\kappa + l_2) > 0$, the derivative of f_m at any of its zeroes has to be negative. As a consequence of the continuity of that function, there exists at most one zero for f_m.

Moreover, as the $\tilde{J}_n(\kappa)$ (62) are easily seen to satisfy the recursive relation

$$4(n-2)\tilde{J}_{n+1}(\kappa) = 2\kappa(2n-3)\tilde{J}_n(\kappa) + (n-1)\tilde{J}_{n-1}(\kappa), \quad n \geq 3, \tag{77}$$

their behaviour at infinity may be proven to be

$$\tilde{J}_n(\kappa) \simeq \beta_n\kappa^{n-3}\left(1+\mathcal{O}\left(\frac{1}{\kappa^2}\right)\right), \quad \beta_n = \frac{\pi(2n-5)!}{[2^{n-2}(n-3)!]^2} \quad \text{when } \kappa \to +\infty,\ n \geq 3,$$

and

$$\tilde{J}_n(\kappa) \simeq \delta_n(-\kappa)^{-n}\left(1+\mathcal{O}\left(\frac{1}{\kappa^2}\right)\right), \quad \delta_n = \frac{[(n-1)!]^2}{(2n-2)!} \quad \text{when } \kappa \to -\infty,\ n \geq 2.$$

As a consequence, $f_m(\kappa)$, positive for $\kappa \leq -l_2$ and going to $-\infty$ when $\kappa \to +\infty$ has one and only one zero: given a positive number l_2, the value of the parameter $\kappa > -l_2$ is uniquely fixed (recall that $l_1 = 0$) and determines, up to a homothety, one and only one nut–nut metric.

4.2.2. Nut–bolt(1) metric

Consider now the situation where the range of x is $]-\infty, x_0]$, $\Omega^2(x_0)=0$, $x_0<\kappa$. Thanks to Lemma 2, the nut at $-\infty$ requires

$$l_1+\tilde{J}_m(\kappa)-2l_2\tilde{J}_{m+1}(\kappa)=0, \tag{78}$$

and, from Lemma 4 and (74), we know that a bolt at x_0 (necessarily a bolt(-1) in order to be compatible with the nut at the other end) implies the two conditions

$$\kappa=x_0+(m-1)\frac{1+x_0^2}{2(x_0+ml_2)},$$
$$l_1(x_0^2-2\kappa x_0-1)+I_m[\kappa,x_0]-2l_2I_{m+1}[\kappa,x_0]=0. \tag{79}$$

Hopefully, for a given value of the parameter l_2, these equations will determine uniquely the other ones (κ, l_1) and fix the metric (66). From (78) and (79) one gets the condition

$$g_m[\kappa,x_0]\equiv[J_m(\kappa,x_0)-\tilde{J}_m(\kappa)]-2l_2[J_{m+1}(\kappa,x_0)-\tilde{J}_{m+1}(\kappa)]=0. \tag{80}$$

Thanks to the increasing character of the function $J_n[\kappa,x]$ as a function of $x<\kappa-\sqrt{1+\kappa^2}$, both square brackets in that equation are positive in that range for x. On the other hand, when $\kappa-\sqrt{1+\kappa^2}<x\le\kappa$, they are both negative. Then, the existence of a solution again requires a strictly positive l_2.

After an integration by parts, $g_m[\kappa,x_0]$ may be rewritten as

$$g_m[\kappa,x_0]=(m-1)\int_{-\infty}^{x_0}\frac{(y+l_2)[\kappa-y]^{m-2}}{[1+y^2]^m}\,\mathrm{d}y+\frac{(m-1)}{2m(1+x_0^2)}\left(\frac{m-1}{2(x_0+ml_2)}\right)^{m-2}. \tag{81}$$

The $J_n(\kappa,x)$ satisfy the following recursion relation ($n\ge 3$):

$$4(n-2)J_{n+1}(\kappa,x)=2\kappa(2n-3)J_n(\kappa,x)+(n-1)J_{n-1}(\kappa,x)$$
$$-\frac{(\kappa-x)^{n-1}}{(x^2-2\kappa x-1)[1+x^2]^{n-2}}, \tag{82}$$

and the same is true for the, positive, square bracket $[J_n(\kappa,x)-\tilde{J}_n(\kappa)]$. As κ and x_0 are related variables, and as from (79) $\kappa-x_0>0$ needs $x_0>-ml_2$, $x_0\to-ml_2^+$ implies $\kappa\to+\infty$, and the behaviour of $[J_n(\kappa,x)-\tilde{J}_n(\kappa)]$ at infinity may be shown to be

$$[J_n(\kappa,x)-\tilde{J}_n(\kappa)]\simeq\gamma_n\kappa^{n-3}\left(1+\mathcal{O}\left(\frac{1}{\kappa}\right)\right)\quad\text{for some positive }\gamma_n>0,\ n\ge 2.$$

As a consequence, $g_m[\kappa,x_0]\simeq-2l_2\gamma_{m+1}\kappa^{m-2}$, goes to $-\infty$ when $x_0\to-ml_2$, $\kappa\to+\infty$. In the same manner, when $x_0\to+\infty$, i.e. $\kappa\simeq\frac{1}{2}(m+1)x_0\to+\infty$, one can prove that $g_m[\kappa,x_0]\simeq-2l_2(-\beta_{m+1}\kappa^{m-2})$. Then it goes to $+\infty$ when $x_0\to+\infty$, $\kappa\to+\infty$.

To sum up, $g_m(\kappa(x_0),x_0)$, varying continuously from $-\infty$ to $+\infty$ when x_0 grows from $-ml_2$ to $+\infty$, has *at least one zero*. We do not succeed in proving that the solution is unique, but our previous results for $n=2m=4$ [25] and computer analysis of the function

$g_m[\kappa(x_0), x_0]$ defined through Eqs. (79) and (81) made us confident on the fact that the parameter $\kappa > -l_2$ *is uniquely fixed* and, due to (78), so is l_1. Finally, given a positive parameter l_2, there is one and only one nut–bolt Einstein–Weyl regular metric.

4.2.3. Bolt(1)–nut metric

Consider now the situation where the range of x is $[x'_0, \kappa]$, $\Omega^2(x'_0) = 0$, $x'_0 < \kappa$. From Lemma 3, the nut at κ requires $l_1 = 0$, and, from Lemma 4 and (74), we know that a bolt at x'_0 (necessarily a bolt(+1) in order to be compatible with the nut at the other end) implies the two conditions

$$\kappa = x'_0 + (m+1)\frac{1 + x'^2_0}{2(-x'_0 + ml_2)},$$
$$l_1(x'^2_0 - 2\kappa x'_0 - 1) + I_m[\kappa, x'_0] - 2l_2 I_{m+1}[\kappa, x'_0] = 0. \tag{83}$$

The same manipulations as in the previous section lead to a similar function

$$g'_m[\kappa, x'_0] \equiv [J_m(\kappa, x'_0) - \tilde{J}_m(\kappa)] - 2l_2[J_{m+1}(\kappa, x'_0) - \tilde{J}_{m+1}(\kappa)]$$
$$= -(m-1)\int_{x'_0}^{\kappa} \frac{(y+l_2)[\kappa - y]^{m-2}}{[1 + y^2]^m}\, \mathrm{d}y + \frac{(m+1)}{2m(1+x'^2_0)}\left(\frac{m+1}{2(-x'_0+ml_2)}\right)^{m-2}. \tag{84}$$

With

$$x_0 = -\frac{(m-1)x'_0 + 2ml_2}{(m+1)}, \tag{85}$$

$\kappa(x'_0)$ of Eq. (83) expressed as a function of x_0, has exactly the same value as $\kappa(x_0)$ of (79). Under the same change of variable, it is shown in Appendix B that the function $g'_m[\kappa, x'_0]$ becomes $-g_m[\kappa, x_0]$ of the previous section (81). Then, except the different values of l_1, the metrics are the same (as discussed for $m = 2$ in [25], only the orientation of the Einstein–Weyl manifold changes).

4.2.4. Bolt(p)–bolt(p) metric

Consider finally the situation where the range of x is $[x'_0, x_0]$, $\Omega^2(x_0) = 0$, $\Omega^2(x'_0) = 0$, $x'_0 < x_0 < \kappa$. From Lemma 4 and (74), we know that a bolt($+p$) at x'_0 and a bolt($-p$) at x_0 imply four relations between κ, x_0, x'_0, l_1 and l_2:

$$\kappa = x'_0 + (m+p)\frac{1 + x'^2_0}{2(-px'_0 + ml_2)} = x_0 + (m-p)\frac{1 + x^2_0}{2(px_0 + ml_2)},$$
$$l_1(x'^2_0 - 2\kappa x'_0 - 1) + I_m[\kappa, x'_0] - 2l_2 I_{m+1}[\kappa, x'_0] = 0,$$
$$l_1(x^2_0 - 2\kappa x_0 - 1) + I_m[\kappa, x_0] - 2l_2 I_{m+1}[\kappa, x_0] = 0. \tag{86}$$

The first two equations, giving a second-order algebraic equation for x_0, lead to two solutions:

$$x_0 = -\frac{(m-p)x'_0 + 2ml_2}{m+p}, \tag{87a}$$

or

$$x_0 = \frac{ml_2 x'_0 + p}{-px'_0 + ml_2}. \tag{87b}$$

The two others, after operations similar to the ones done in the two previous sections, lead to the vanishing of a new function:

$$\begin{aligned} h_m[x_0, x'_0] &\equiv [J_m(\kappa, x_0) - J_m(\kappa, x'_0)] - 2l_2[J_{m+1}(\kappa, x_0) - J_{m+1}(\kappa, x'_0)] \\ &= (m-1)\int_{x'_0}^{x_0} \frac{(y+l_2)[\kappa - y]^{m-2}}{[1+y^2]^m}\, \mathrm{d}y + \frac{m-p}{2m(1+x_0^2)} \left(\frac{m-p}{2(px_0 + ml_2)}\right)^{m-2} \\ &\quad - \frac{m+p}{2m(1+x_0'^2)} \left(\frac{m+p}{2(-px'_0 + ml_2)}\right)^{m-2}. \end{aligned} \tag{88}$$

Of course, here also one finds no solution when $l_2 \le 0$. We shall first prove that if $p \ge m$, there is no solution, second that for $p < m$ relation (87b) is excluded.

- *Case* $p = m$. As $x_0 \ne \kappa$, Eq. (86) enforces $x_0 = -l_2$, and x'_0 is related to κ by $\kappa = (l_2 x'_0 + 1)/(l_2 - x'_0)$. The function $h_m[-l_2, x'_0]$ reduces itself to the sum of two strictly negative terms (the quotient $(m-p)/2(px_0 + ml_2) \equiv (\kappa - x_0)/(1 + x_0^2) = 1/(l_2 - x'_0)$ being finite as $x'_0 < x_0 = -l_2$). So there is no bolt(m)–bolt(m) Einstein–Weyl metric.
- *Case* $p > m$. One has $x'_0 < x_0 < -ml_2/p$. This condition is readily seen to contradict solution (87b) and one is left with solution (87a). Then, the positivity of $x_0 - x'_0 = -2m(x'_0 + l_2)/(p - m)$ enforces $x'_0 < -l_2$, and the relation $x_0 + l_2 = ((p - m)/(p + m))(x'_0 + l_2)$ also ensures that $x_0 < -l_2$. As a consequence, the function $h_m[x_0, x'_0]$ reduces itself to the sum of three strictly negative terms and there are no bolt(p)–bolt(p) Einstein–Weyl metric for $p > m$. Then we have the following lemma.

Lemma 5. *Regular bolt–bolt Einstein–Weyl SU(m)-invariant metrics, non-conformally Einstein, may exist only with a twist* $p < m$.

Note that this was only conjectured in [47].

- *Case* $p < m$. Consider first the candidate solution (87b). Using relations (86) and some identities

$$\frac{px_0 + ml_2}{1 + x_0^2} = \frac{-px'_0 + ml_2}{1 + x_0'^2}, \qquad 1 + x_0 x'_0 = ml_2 \frac{1 + x_0^2}{px_0 + ml_2},$$
$$(\kappa - x_0) + (\kappa - x'_0) = m\frac{1 + x_0^2}{px_0 + ml_2},$$

  one may rewrite the function h_m as

$$h_m[x_0, x'_0] = -\left(\frac{m-1}{m}\right)\left(\frac{px_0 + ml_2}{1 + x_0^2}\right)\int_{x'_0}^{x_0} \frac{(y - x'_0)(x_0 - y)[\kappa - y]^{m-2}}{[1+y^2]^m}\, \mathrm{d}y, \tag{89}$$

  whose negatively definite property ensures that there is no "solution (87b)" candidate.

Then one is left with $p < m$ and the linear relation (87a) $x_0 = -((m-p)x_0'+2ml_2)/(m+p)$ between x_0 and x_0'. Some useful identities result from the previous relation:

$$0 < x_0 - x_0' = -\frac{2m}{m+p}(x_0' + l_2) = \frac{2m}{m-p}(x_0 + l_2),$$

and imply

$$x_0 > -l_2 > -\frac{ml_2}{p}, \qquad x_0' < -l_2 < \frac{ml_2}{p}.$$

One then obtains the $x_0' \to -\infty,\ x_0 \to +\infty$ limit of $h_m[x_0, x_0']$ to be $+\infty$:

$$h_m[x_0, x_0'] \simeq -2l_2[-2\beta_{m+1}(\kappa)^{m-2}], \quad x_0' \to -\infty,\ x_0 \to +\infty. \tag{90}$$

With regards to the limit $x_0' \to -l_2,\ x_0 \to -l_2$, one gets

$$h_m[-l_2, -l_2] = -\frac{p}{m(1+l_2^2)^2(2l_2)^{m-2}} < 0. \tag{91}$$

Then there exists *at least one zero* of $h_m[x_0(x_0'), x_0']$. We do not succeed in proving that the solution is unique, but our previous results for $n = 2m = 4$ [25] and computer analysis of the function $h_m[x_0(x_0'), x_0']$ defined through Eq. (88) made us confident that the parameter $\kappa > -l_2$ is uniquely fixed and, due to (86), so is l_1. Finally, given a positive parameter l_2, there is one and only one bolt–bolt Einstein–Weyl regular metric, and we have the following lemma.

Lemma 6. *Regular bolt–bolt Einstein–Weyl $SU(m)$-invariant metrics, non-conformally Einstein, exist for any twist $p < m$, and depend on a single positive parameter l_2.*

Note also that relation (87a) implies

$$\frac{\kappa - x_0}{1 + x_0^2} = \frac{\kappa - x_0'}{1 + x_0'^2} \Leftrightarrow \omega_3[x_0] = \omega_3[x_0']. \tag{92}$$

4.3. Summary

In our Gauduchon gauge, we found $m + 2$, and only $m + 2$, families of non-conformally Einstein regular Einstein–Weyl $SU(m)$-invariant metrics ; according to the classification of Gibbons and Hawking, they are complete and live on a compact orientable manifold without boundary.

The same analysis with two functions of T could have been done for any other $(n-2)$-dimensional symmetric Kähler space with little changes, e.g. for the Grassmannian $SU(p+q)/(SU(p) \times SU(q) \times U(1))$, with $pq = \frac{1}{2}(n-2)$.

5. $S^1 \times SO(n-1)$-invariant structures

Cohomogeneity-one Weyl structures (48) with $S^1 \times SO(n-1)$ invariance may be written in a Gauduchon gauge as (here, thanks to (49), $\mathrm{d}y^0 = 0 \Rightarrow y^0 = \mathrm{d}\theta$) [31]

$$\mathrm{d}s^2 = (\mathrm{d}T)^2 + f^2(T)(\mathrm{d}\theta)^2 + h^2(T)g_{\mathrm{B}}, \qquad \gamma = \pm \Gamma f^2(T)\,\mathrm{d}\theta, \quad \theta \in (0, 2\pi), \tag{93}$$

where g_{B} is the standard metric on S^{n-2} with Ricci curvature $= (n-3)g_{\mathrm{B}}$. Note that an exact structure exists iff $f^2(T) = \text{constant}$.

5.1. Local expressions

The Einstein–Weyl equation (46) write [31,32]

$$\begin{aligned} &\Lambda' = -\frac{f''}{f} - (n-2)\frac{h''}{h} - \frac{1}{2}\Gamma^2 f^2, \\ c_{(00)}: \quad &\Lambda' = -\frac{f''}{f} - (n-2)\frac{h'f'}{hf} + \frac{n-4}{4}\Gamma^2 f^2, \\ c_{(ij)}: \quad &\Lambda' = -\frac{h''}{h} - (n-3)\frac{h'^2}{h^2} - \frac{h'f'}{hf} + \frac{n-3}{h^2} - \frac{1}{2}\Gamma^2 f^2. \end{aligned} \tag{94}$$

Note that an exact structure solution exists and writes

$$\mathrm{d}s^2 = \frac{4f^2}{\Gamma^2}\left[(\mathrm{d}t')^2 + \sin^2 t' g_{\mathrm{B}} + \frac{\Gamma^2}{4}(\mathrm{d}\theta)^2\right], \quad \gamma = \pm \Gamma f^2\,\mathrm{d}\theta,$$

the metric is the standard metric on $S^1 \times S^{n-1}$.

Here again, we rewrite (93) with notations inspired by gravitation [48,49]

$$\mathrm{d}s^2 = \left[\omega^2(t)\omega_3(t)(\mathrm{d}t)^2 + \frac{\omega^2(t)}{\omega_3(t)}(\mathrm{d}\theta)^2\right] + \omega_3(t)(\mathrm{d}\tau)^2, \quad \gamma = \pm \Gamma \frac{\omega^2(t)}{\omega_3(t)}\,\mathrm{d}\theta, \tag{95}$$

and define $u(t)$ through

$$u(t) = \frac{1}{\omega_3\omega^2}\frac{\mathrm{d}\omega_3}{\mathrm{d}t}. \tag{96}$$

The difference of the first two equations (94) allows the calculation of the derivative of $u(t)$ which is found to have the same expression as in Section 4.1:

$$\frac{\mathrm{d}u}{\mathrm{d}t} = -\frac{1}{2}\omega^2[\Gamma^2 + u^2] < 0.$$

Then, one can change the variable t into u and compute

$$\frac{\mathrm{d}\omega_3}{\mathrm{d}u} = -2\frac{u\omega_3}{\Gamma^2 + u^2},$$

which integrates to

$$\omega_3(u) = \frac{2k}{\Gamma^2 + u^2}, \quad k > 0 \text{ thanks to positivity.} \tag{97}$$

Defining

$$\Omega^2 = \tfrac{1}{4}(\Gamma^2 + u^2)\omega^2, \tag{98}$$

and using the Einstein–Weyl equations (94), after a rescaling of u and k according to $u = \Gamma x,\ k = \Gamma\kappa$, one obtains a second-order linear differential equation

$$(1+x^2)\frac{\mathrm{d}^2\Omega^2}{\mathrm{d}x^2} - (n-6)x\frac{\mathrm{d}\Omega^2}{\mathrm{d}x} - 2(n-3)[\Omega^2 - 1] = 0. \tag{99}$$

It solves to

$$\Omega^2(x) = 1 - l_1 x(1+x^2)^{(n-4)/2} - l_2[1 + (n-3)x(1+x^2)^{(n-4)/2}K_n(x)], \tag{100}$$

where ($n \geq 3$)

$$K_n(x) = \int_0^x \frac{\mathrm{d}y}{(1+y^2)^{(n-2)/2}}. \tag{101}$$

For further use, notice that when $x \to \pm\infty$, the functions $K_n(x)$ behave as ($n \geq 4$)

$$K_n(x) \simeq \pm\left[a_n + \frac{1}{(n-3)|x|^{n-3}}\right],\quad x \to \pm\infty,\ a_n = \frac{\Gamma[(n-3)/2]\Gamma[1/2]}{2\Gamma[(n-2)/2]}. \tag{102}$$

Eqs. (97) and (100) and

$$\frac{\mathrm{d}u}{\mathrm{d}t} = -2\Omega^2 \tag{103}$$

give the distance [11] and Weyl-form as functions of the new "proper time" x:

$$\mathrm{d}s^2 = \frac{2\kappa}{\Gamma}\left[\frac{(\mathrm{d}x)^2}{\Omega^2(x)(1+x^2)^2} + \frac{\Omega^2(x)}{\kappa^2}(\mathrm{d}\theta)^2 + \frac{(\mathrm{d}\tau)^2}{(1+x^2)}\right], \qquad \gamma = \pm\frac{2\Omega^2(x)}{\kappa}\mathrm{d}\theta. \tag{104}$$

For further reference, note that the positive parameter κ only appears in the combination $\mathrm{d}\theta/\kappa$, and as a rescaling of the homothety parameter Γ.

The distance may be rewritten as a function of the angle $\Psi \in [0, \pi],\ \cot\Psi = x$,

$$\mathrm{d}s^2 = \frac{2\kappa}{\Gamma}\left[\frac{(\mathrm{d}\Psi)^2}{\Omega^2(\Psi)} + \frac{\Omega^2(\Psi)}{\kappa^2}(\mathrm{d}\theta)^2 + \sin^2\Psi(\mathrm{d}\tau)^2\right],$$
$$\Omega^2(\Psi) = 1 - l_2 - \cos\Psi\,\sin^{3-n}\Psi\left[l_1 + (n-3)l_2\int_\Psi^{\pi/2}\sin^{n-4}\phi\,\mathrm{d}\phi\right]. \tag{105}$$

Notice that for $n = 3$, the differential equation (99) solves to $\Omega^2 = 1 - l_2 - l_1\cos\Psi$, in agreement with (105): $\Omega^2(\Psi)$ varies monotonically between $1 - l_1 - l_2$ and $1 + l_1 - l_2$, then it has at most one zero.

[11] Of course, the parameters κ, l_1, l_2 and the proper time x are constrained by positivity $\Omega^2 > 0,\ \kappa > 0$.

Finally, the conformal scalar curvature is computed from (47) and (94):

$$S^D = \frac{\Gamma}{2\kappa}[nl_2 + n(n-4)(1-\Omega^2(x))] \leq \frac{\Gamma}{2\kappa} n(l_2 + n - 4). \tag{106}$$

Note that a constant conformal scalar curvature requires either $n = 4$ or $\Omega^2(x) = 1$. This last case corresponds to an exact Weyl-form (note that in our local approach, a closed Weyl-form is an exact one) and the metric (105) is the standard metric on $S^1 \times S^{n-1}$ [33]. Then, we have proved the following theorem. [12]

Theorem 4. *The most general* ($n \geq 4$)*-dimensional* (*non-*) *compact non-exact Einstein–Weyl structure with an* $S^1 \times SO(n-1)$*-invariant metric is a 3-parameter structure* (*plus one homothetic parameter*).

The metric has a constant conformal curvature in the Gauduchon gauge if and only if the dimension $n = 4$.

In the following section, we shall consider the possible positive definite and regular $S^1 \times SO(n-1)$-invariant Einstein–Weyl metrics, still with the tools of nuts and bolts. We shall prove that, up to an arbitrary homothetic factor $\Gamma > 0$, there exist three one-parameter families of complete Einstein–Weyl metrics with a non-exact Weyl form, depending on a strictly positive constant l_2 related to the conformal scalar curvature.

5.2. Regular metrics

The function $\Omega^2(x)$ has to be positive on the proper time interval. The possible singularities of the distance occur at $x = \pm\infty$, or at a zero of the function $\Omega^2(x)$. The case $n = 3$, which requires a special analysis as the candidates are not solely given by the ansatz (93) [28], will not be considered in the following.

(a) *Regularity of the distance as* $x \to \pm\infty$. When $x \to \pm\infty$, $\Omega^2(x) \simeq -\delta_n^{\pm}|x|^{n-3}$ where $\delta_n^{\pm} = l_1 \pm (n-3)a_n l_2$. As above, the behaviour of the distance is readily seen to be singular if $\delta_n^{\pm} \neq 0$.

Consider now the special cases when $\delta_n^{\pm}$ vanishes: thanks to (102), the function $\Omega^2(x)$ goes to 1 when $x \to \pm\infty$. So, the distance behaves as

$$ds^2 \sim \frac{2\kappa}{\Gamma}\left[\frac{(dx)^2}{(x)^4} + \frac{1}{\kappa^2}(d\theta)^2 + \frac{1}{x^2}(d\tau)^2\right]. \tag{107}$$

Under the change $\rho = 1/x$:

$$ds^2 \simeq \frac{2\kappa}{\Gamma}\left[(d\rho)^2 + \rho^2(d\tau)^2 + \frac{1}{\kappa^2}(d\theta)^2\right], \quad \rho \to 0.$$

[12] For $n = 3$, the ansatz (93) corresponds to the special case $f = 0$ of Tod's general analysis on three-dimensional Einstein–Weyl structures [28]: his four parameters (f, λ, B and C) may, respectively, be rewritten as $f = 0$, $\lambda = \Gamma$, $B = (1-l_2)\Gamma/4\kappa$ and $C = [(l_1)^2-(1-l_2)^2]/4\kappa^2$; his co-ordinates are, respectively, $V = \sqrt{2\Omega^2(x)/\kappa\Gamma}$, $t = \theta$ and $(dy) = \sqrt{8\kappa^3/\gamma l_1^2}(d\tau)$.

The singularity is removable if one changes to Cartesian co-ordinates in the $(n-1)$-dimensional space: near the end point $\rho = 0$, the manifold is a circle S^1 which, generalising Gibbons and Hawking terminology [34,35], we call a bolt(S^1). To sum up, we have the following lemma.

Lemma 7. *If the proper time interval extends to* $\pm\infty$, *the metric can be regular only if* $\delta_n^{\pm} \equiv l_1 + l_2(n-3)K_n(\pm\infty) = 0$, *and then a bolt* (S^1) *occurs.*

A corollary is that the sole solution with $]-\infty, +\infty[$ as proper time interval, requires $l_1 = l_2 = 0$ i.e. $\Omega^2(x) = 1$ which leads to the metric on the S^{n-1} sphere.

(b) *Regularity of the distance at a zero of* $\Omega^2(x)$. If $\Omega^2(x_0) = 0$ with $(\mathrm{d}\Omega^2/\mathrm{d}x)(x_0) = 0$, the differential equation (99) enforces x_0 to be a maximum, which contradicts positivity. Then, change the variable x to ρ according to

$$x = x_0 + \rho^2 \frac{\mathrm{d}\Omega^2}{\mathrm{d}x}(x_0), \tag{108}$$

using $\Omega^2(x) \simeq \rho^2[(\mathrm{d}\Omega^2/\mathrm{d}x)(x_0)]^2$, the distance behaves when $\rho \to 0$ as

$$\mathrm{d}s^2 \simeq \frac{8\kappa}{\Gamma[1+x_0^2]^2}\left[(\mathrm{d}\rho)^2 + \rho^2\left(\left(\frac{1+x_0^2}{2\kappa}\right)\frac{\mathrm{d}\Omega^2}{\mathrm{d}x}(x_0)\right)^2(\mathrm{d}\theta)^2 + \frac{1+x_0^2}{4}(\mathrm{d}\tau)^2\right], \quad \rho \to 0. \tag{109}$$

If

$$\frac{(1+x_0^2)}{2\kappa}\frac{\mathrm{d}\Omega^2}{\mathrm{d}x}(x_0) = \pm p, \quad p = 1, 2, \ldots,$$

the singularity is removable if one changes to Cartesian co-ordinates in the two-dimensional space (ρ, θ), and restricts the range in the angle θ to the interval $[0, 2\pi/p]$: near the end point $\rho = 0$, the manifold is the sphere S^{n-2} which gives a bolt [34,35]. As was previously remarked, the integer p that "divide" the θ interval, may be reabsorbed into the definition of the parameters κ and Γ: so, without loss of generality, we shall only consider $p = 1$. To sum up, we have the following lemma.

Lemma 8. *If the function* $\Omega^2(x)$ *vanishes at* x_0, *the metric can be regular only if* x_0 *is a bolt* (S^{n-2}) *of twist* 1, *i.e.*

$$\Omega^2(x_0) = 0, \qquad \left(\frac{1+x_0^2}{2\kappa}\right)\frac{\mathrm{d}\Omega^2}{\mathrm{d}x}(x_0) = \pm 1. \tag{110}$$

If $(\mathrm{d}\Omega^2/\mathrm{d}x)(x_0) > 0$ the bolt is $(+1)$ and the proper time interval extends up to $+\infty$ or to another bolt(-1) at $x_1 > x_0$; on the other situation, the bolt at x_0 is a (-1) one and the proper time interval extends down to $-\infty$ or to another bolt$(+1)$ at $x_2 < x_0$.

Condition (110) may be rewritten as a relation between κ, l_2 and x_0:

$$\kappa = \begin{cases} \mp\dfrac{1-l_2+(n-3)x_0^2}{2x_0} & \text{if } x_0 \neq 0, \\ \dfrac{(n-3)a_n}{2} & \text{if } x_0 = 0 \Leftrightarrow l_2 = 1. \end{cases} \tag{111}$$

We have now all the building blocks needed in our discussion on the regularity of our Einstein–Weyl metrics, according to the possible proper time intervals.

5.2.1. Bolt(S^1)–bolt(S^1) metric

Consider a situation where the allowed range of x is the largest one $]-\infty, +\infty[$. According to Lemma 7, $\Omega^2(x) = 1$ and the Einstein–Weyl structure is an exact one, conformal to the Einstein case $S^1 \times S^{n-1}$. Then we are not interested.

5.2.2. Bolt(S^{n-2})–bolt(S^1) metric

Consider now a situation where the allowed range of x is $[x_0, +\infty[$ with $\Omega^2(x_0) = 0$. Thanks to Lemma 7,

$$l_1 = -(n-3)a_n l_2,$$

and, from Lemma 8 and (111), we know that a bolt(S^{n-2}) at x_0 implies the two conditions

$$\kappa = \begin{cases} -\dfrac{1-l_2+(n-3)x_0^2}{2x_0}, & x_0 \neq 0, \\ \dfrac{(n-3)a_n}{2}, & x_0 = 0 \Leftrightarrow l_2 = 1, \end{cases} \quad \text{and} \quad \Omega^2(x_0) = 0, \tag{112}$$

The derivative of Ω^2 may be written as

$$\begin{aligned} \frac{d\Omega^2}{dx} &= (n-3)l_2[1+(n-3)x^2](1+x^2)^{(n-6)/2}G(x), \\ G(x) &= -\frac{x}{[1+(n-3)x^2](1+x^2)^{(n-4)/2}} + \int_x^\infty \frac{dy}{(1+y^2)^{(n-2)/2}}, \\ \frac{dG}{dx} &= -\frac{2(1+x^2)}{[1+(n-3)x^2]^2(1+x^2)^{(n-2)/2}} < 0. \end{aligned} \tag{113}$$

$G(x)$, decreasing from $2a_n$ to 0 between $x = -\infty$ and $+\infty$, is strictly positive ($G(0) = a_n$). As a consequence, if $l_2 \leq 0$, $\Omega^2(x)$ monotonically decreases from $+\infty$ to 1 and cannot vanish. On the contrary, if $l_2 > 0$, $\Omega^2(x)$ monotonically increases from $-\infty$ to 1 and its vanishing determines a unique value for the parameter x_0.

Notice also that in the range $[x_0, +\infty[$,

$$0 \leq \Omega^2(x) < 1 \Rightarrow l_2\frac{n\Gamma}{2\kappa} \leq S^D \leq (l_2+(n-4))\frac{n\Gamma}{2\kappa}, \tag{114}$$

the conformal scalar curvature is a strictly positive function on the manifold, whatever the dimension $n \geq 4$, be, in agreement with a theorem of Calderbank for the compact case [33].

To summarise, given a positive parameter l_2, l_1 and κ are fixed, and, up to an homothety, there is one and only one $S^{n-2} - S^1$ Einstein–Weyl regular metric. Its scalar conformal curvature is a strictly positive function on the manifold. The particular case $l_2 = 1$ requires $x_0 = 0$, $l_1 = -(n-3)a_n$ and $\kappa = \frac{1}{2}(n-3)a_n$.

5.2.3. *Bolt(S^1)–bolt(S^{n-2}) metric*

Consider now a situation where the allowed range of x is $]-\infty, x_0']$ with $\Omega^2(x_0') = 0$. The same discussion as in the previous section ($\Omega^2(x)$ is unchanged when $x \to -x$ and $l_1 \to -l_1$) gives a unique solution for x_0' for any $l_2 > 0$ ($x_0' = -x_0$ of the previous section). The other parameters are fixed:

$$l_1 = (n-3)a_n l_2, \qquad \kappa = \frac{1 - l_2 + (n-3)x_0'^2}{2x_0'}.$$

As $G(x)$ of (113) is changed into $G(x) - 2a_n$ which is <0, now $\Omega^2(x)$ monotonically decreases from 1 to 0. Here again, up to an homothety, there is one and only one $S^1 - S^{n-2}$ Einstein–Weyl regular metric, still with a positive scalar conformal curvature. The metrics are the same, only the orientation of the Einstein–Weyl manifold changes. The particular case $l_2 = 1$ requires $x_0 = 0$, $l_1 = (n-3)a_n$ and $\kappa = \frac{1}{2}(n-3)a_n$.

5.2.4. *Bolt(S^{n-2})–bolt(S^{n-2}) metric*

Consider finally a situation where the allowed range of x is $[x_0', x_0]$. From Lemma 8 and (111), we know that a bolt(+1) at x_0' and a bolt(−1) at x_0 imply four relations between κ, x_0, x_0', l_1 and l_2:

$$\kappa = -\frac{1 - l_2 + (n-3)x_0'^2}{2x_0'} = \frac{1 - l_2 + (n-3)x_0^2}{2x_0}, \qquad l_2 \neq 1,$$
$$0 = \Omega^2(x_0') = \Omega^2(x_0). \tag{115}$$

(The case $l_2 = 1$ is excluded as the last two equations (115) imply: $l_1 + (n-3)K_n(x_0') = l_1 + (n-3)K_n(x_0) = 0$ which enforces $x_0 = x_0' = l_1 = 0$ which is forbidden!) The first two equations lead to two possibilities:

$$x_0 = -x_0', \tag{116a}$$

or

$$x_0 x_0' = \frac{l_2 - 1}{n-3}. \tag{116b}$$

Eliminating l_1 between the two others leads to the vanishing of a new function

$$h_n[x_0, x_0'] \equiv [\alpha_n(x_0) - \alpha_n(x_0')] - (n-3)\frac{l_2}{1-l_2}[K_n(x_0) - K_n(x_0')] = 0,$$
$$\text{with} \qquad \alpha_n(x) = \frac{1}{x(1+x^2)^{(n-4)/2}}. \tag{117}$$

The second square bracket in (117) is positive. The function α_n is monotonically decreasing in the two domains $x < 0$ and $x > 0$. So, if x_0 and x'_0 have the same sign, $l_2/(1-l_2)$ has to be negative; on the other case, the first square bracket in (117) is positive and $l_2/(1-l_2)$ has to be positive. Then, solution (116a) requires $0 < l_2 < 1$ and solution (116b) either $l_2 > 1$, the two zeroes of Ω^2 being of the same sign, or $0 < l_2 < 1$ when they are of opposite sign. In both cases, $l_2 \leq 0$ is excluded.

- *Solution* (116b). The derivative of the function $h_n[x_0, x'_0(x_0)]$ is

$$\frac{\mathrm{d}h_n[x_0, x'_0(x_0)]}{\mathrm{d}x_0} = -\left(\frac{1}{x_0^2} + \frac{n-3}{1-l_2}\right)\left[\frac{1}{(1+x_0^2)^{(n-2)/2}} - \frac{1}{(1+x_0'^2)^{(n-2)/2}}\right].$$

  - $l_2 > 1$: x_0 and x'_0 have the same sign (as $(h_n[x_0, x'_0(x_0)]$ is an odd-parity function, we may chose a positive sign), then $x_0 > \sqrt{(l_2-1)/(n-3)}$ and $\mathrm{d}h_n[x_0, x'_0(x_0)]/\mathrm{d}x_0$ is negative: as a consequence, $h_n[x_0, x'_0(x_0)]$ decreasing from 0 when $x_0 = \sqrt{(l_2-1)/(n-3)}$ to $-\infty$ when x_0 goes to $+\infty$ does not vanish.
  - $0 < l_2 < 1$: x_0 positive, and $h_n[x_0, x'_0(x_0)]$ has a minimum for $x_0 = -x'_0 = \sqrt{(1-l_2)/(n-3)}$ which is shown to be positive when $l_2 \in]0, 1[$, that also excludes any solution to (117).

  Then we are left with case (116a).
- *Solution* (116a). $l_1 = 0$ results from the difference of the last two equations (115) with $x_0 = -x'_0$. Moreover, with $0 < l_2 < 1$,

$$\frac{\mathrm{d}h_n[x_0, -x_0]}{\mathrm{d}x_0} = -2\frac{1-l_2+(n-3)x_0^2}{(1-l_2)x_0^2(1+x_0^2)^{(n-2)/2}}$$

  is negative and $h_n[x_0, -x_0]$, decreasing from $+\infty$ to $-2l_2(n-3)a_n/(1-l_2)$ when x_0 goes from 0 to $+\infty$, has a unique zero x_0.

To sum up, given a positive parameter $l_2 < 1$, there is one and only one bolt(+1)–bolt(−1) Einstein–Weyl regular metric with $l_1 = 0$ and $\kappa = (1 - l_2 + (n-3)x_0^2)/2x_0$.

Note also that relation (116a) implies

$$\omega_3[x_0] = \omega_3[x'_0]. \tag{118}$$

Moreover, as now

$$\frac{\mathrm{d}\Omega^2}{\mathrm{d}x} = (n-3)l_2[1+(n-3)x^2](1+x^2)^{(n-6)/2}[G(x) - a_n] \tag{119}$$

decreases from a_n to $-a_n$, $\Omega^2(x)$ has a single maximum between $-x_0$ and x_0, precisely at $x = 0$ as $G(0) = a_n$. As a consequence

$$0 \leq \Omega^2(x) \leq (1-l_2) < 1 \Rightarrow (n-3)l_2\frac{n\Gamma}{2\kappa} \leq S^D \leq (l_2 + (n-4))\frac{n\Gamma}{2\kappa} \tag{120}$$

is positive on the manifold, whatever the dimension $n \geq 4$.

5.3. Summary

In our Gauduchon gauge, we found three, and only three, families of non-conformally Einstein regular Einstein–Weyl metrics; according to the classification of Gibbons and Hawking, they are complete and live on a compact manifold without boundary; moreover, they have a positive conformal scalar curvature in agreement with the theorem of Calderbank [33].

6. Concluding remarks

In this paper, we have first presented a local analysis of n-dimensional Einstein–Weyl structures (g, γ) corresponding to cohomogeneity-one metrics in a Gauduchon gauge. Second, we have discussed with some details the explicit solutions in the case of an $SU(m)$ group of left-isometries and in the case of an $S^1 \times SO(n-1)$ group.

In the first part, we emphasised the role of the extra isometry exhibited by Tod, we explicated its action for cohomogeneity-one structures, and we gave a necessary condition for the existence of a non-exact Einstein–Weyl structure (Lemma 1); moreover, for a large subclass, we proved that the metric is locally conformally Kähler (Theorem 2).

In the second part, we presented a complete local analysis of the two families, we showed that they depend on three arbitrary parameters (plus one homothetic one), we gave the Kähler form (for a conformally related metric) for the first case; then, in both cases, we analysed the consequences of the completeness requirement and we obtained one-parameter families of solutions (plus one homethetic parameter $\Gamma > 0$).

Let us finally compare our results with previous ones. Of course, they are not new when compared to global mathematical approaches, but here we mainly required only local properties and so we obtained all the local solutions. We also used a language more relevant for physicists and, as in the search for special solutions we found a simpler parameterisation, we were able to prove the conjectures in [31] and to correct some mistakes in the four-dimensional analysis of [32].

- As the analysis of Gibbons and Hawking in the language of bolts and nuts applies only to orientable manifolds, it is not surprising that we missed metrics on non-orientable manifolds such as RP^4 or $RP^4 \# CP^2$, contrary to [31,32].
- There is a correspondence between nuts and bolts à la Gibbons and Hawking [34,35] and special orbits in the language of mathematicians:
  - a nut corresponds to special orbit being a point,
  - a bolt(p) in $n = 2m$ dimensions, corresponds to special orbit being $\mathbb{C}P^{m-1}$; the integer p $(p < m)$ means that the original $(n-1)$-dimensional homogeneous space $SU(m)/SU(m-1)$ has been restricted, through Einstein–Weyl constraints and regularity requirements, to $((U(1)/\mathbb{Z}_p) \times SU(m))/U(m-1)$,
  - a bolt(S^1) corresponds to special orbit being a circle,
  - a bolt(S^{n-2}) corresponds to special orbit being an $(n-2)$-dimensional sphere.

- Our nut–bolt families (Sections 4.2.2 and 4.2.3) correspond to the same manifold $\mathbb{C}P^m$ with both orientations: so the solutions are not really different solutions. The same remark also holds for the bolt(S^1)-bolt(S^{n-2}) solutions of Sections 5.2.2 and 5.2.3, the manifold being S^n.
- Our bolt–bolt families (Sections 4.2.4 and 5.2.4) corresponds to Madsen's ones [31, Sections 8.24 and 7.40], but we have been able to prove that for structures of cohomogeneity-one under $SU(m)$, no bolt(p)–bolt(p) exists with $p \geq m$ (Lemma 5), a result which was only conjectured. Moreover, thanks to our parameterisation that disentangles the parameters κ, x_0 and x'_0 into a single transcendental equation for only one unknown parameter, we also proved that the relation conjectured in Madsen's thesis dissertation (the "time parameter" in these analyses being an angle φ) : $\Phi_1 + \Phi_2 = \pi \Leftrightarrow h^2(T_0) = h^2(T'_0) \Leftrightarrow \omega_3(x_0) = \omega_3(x'_0)$ (cf. (92) and (118)), is indeed the sole solution, for any $n \geq 4$.

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Appendix A. Cohomogeneity-one geometry

The n-dimensional distance being split into

$$\mathrm{d}s^2 = (\mathrm{d}T)^2 + h_{ij}[T]e^i e^j = (\mathrm{d}T)^2 + g_{\alpha\beta}\,\mathrm{d}x^\alpha\,\mathrm{d}x^\beta,$$

and using the quantities K_i^j given in (26), the Christoffel connection components are expressed as

$$\begin{aligned}&2\Gamma^\alpha_{0\beta} = E_i^\alpha e_\beta^j K_j^i, \qquad 2\Gamma^0_{\alpha\beta} = -e^i_\alpha e^j_\beta K_{ij},\\ &2\Gamma^\alpha_{\beta\gamma} = g^{\alpha\delta}[g_{\delta\beta,\gamma} + g_{\delta\gamma,\beta} - g_{\beta\gamma,\delta}], \text{ the other components vanishing,}\\ &\text{with } g^{\alpha\beta}g_{\beta\gamma} = \delta^\alpha_\gamma, \qquad E_i^\alpha e_\alpha^j = \delta_i^j, \qquad E_i^\alpha e^i_\beta = \delta^\alpha_\beta.\end{aligned} \tag{A.1}$$

The covariant derivative of the vielbeins e^i_α is readily computed:

$$\nabla_\beta e^i_\alpha = \partial_\beta e^i_\alpha - \Gamma^\gamma_{\beta\alpha}e^i_\gamma = \partial_\beta e^i_\alpha - \tfrac{1}{2}h^{il}h_{mn}E_l^\gamma[\partial_\beta(e^m_\alpha e^n_\gamma) + \partial_\alpha(e^m_\beta e^n_\gamma) - \partial_\gamma(e^m_\alpha e^n_\beta)], \tag{A.2}$$

which, using (17a), (17b) and (19), reduces to

$$\nabla_\beta e^i_\alpha = -[\tfrac{1}{2}f^i_{jk}e^j_\beta + f^i_{ak}\omega^a_\beta]e^k_\alpha + h^{ij}h_{k(l}f^k_{n)j}e^l_\alpha e^n_\beta. \tag{A.3}$$

A related result is

$$\nabla_\alpha E_i^\alpha = f^k_{ki} + \omega^a_\beta E_k^\beta f^k_{ai}. \tag{A.4}$$

With (A.1), the n-dimensional Ricci tensor is expressed in function of the tensor $h_{ij}[T]$, its derivative K_{ij} and the $(n-1)$-dimensional Ricci tensor (26). The expression

$$2R^{(\nabla)}_{0\alpha} = \nabla_\beta(e^i_\alpha E^\beta_j K^j_i) - \nabla_\alpha(e^i_\beta E^\beta_j K^j_i)$$

simplifies to

$$K^j_i[T]\nabla_\beta(e^i_\alpha E^\beta_j),$$

which, using (17a), (17b) and (A.3), reduces to

$$2R^{(\nabla)}_{0\alpha} = e^i_\alpha[K^j_i f^k_{kj} + K^j_k f^k_{ij}]. \tag{A.5}$$

A.1. The Bianchi identity

The $\nu = 0$ component of the Bianchi identity $2\nabla_\mu R^{(\nabla)\mu}_\nu = \nabla_\nu R^{(\nabla)}$ is split according to $\mu = (0, \alpha)$. Using (26), (A.2) and

$$R^{(\nabla)} = R^{(n-1)} + 2R^{(\nabla)}_{00} - \frac{1}{4}\left(\frac{h'}{h}\right)^2 + \frac{1}{4}K_{ij}K^{ij},$$

one obtains

$$2\nabla_\mu R^{(\nabla)\mu}_0 = \nabla_0 R^{(\nabla)} - h^{ij}\frac{\mathrm{d}R^{(n-1)}_{ij}}{\mathrm{d}T} + 2\nabla^{(n-1)}_\alpha R^{(\nabla)\alpha}_0.$$

As a consequence

$$h^{ij}\frac{\mathrm{d}R^{(n-1)}_{ij}}{\mathrm{d}T} = 2[\nabla_\alpha E^\alpha_k]R^i_0,$$

where, with (A.5),

$$2R^i_0 = 2h^{ij}E^\alpha_j R^{(\nabla)}_{0\alpha} = K^{ji} f^k_{kj} + K^j_k h^{il} f^k_{lj}.$$

Appendix B. Bolt–nut versus nut–bolt

The relation (87a)) whose particular case is (85) implies

$$\frac{x_0 + x'_0}{1 - x_0 x'_0} = \frac{-1}{\kappa}$$

(κ being related to x_0 and x'_0 through (87a) and (87b)). With

$$\phi_0 = \tan^{-1}(x_0), \qquad \phi'_0 = \tan^{-1}(x'_0), \qquad \psi = \tan^{-1}(\kappa),$$
$$\phi_0, \phi'_0 \in]-\pi/2, +\pi/2[,\ \psi \in]\phi_0, +\pi/2[,$$

this relation writes

$$\psi - \phi_0 = \frac{\pi}{2} + \phi'_0. \tag{B.1}$$

The following identity

$$H_n(\kappa, x_0) \equiv \int_{-\infty}^{x_0} \frac{[\kappa - y]^{n-1}}{(1+y^2)^n}\,\mathrm{d}y = \int_{x_0'}^{\kappa} \frac{[\kappa - z]^{n-1}}{(1+z^2)^n}\,\mathrm{d}z \equiv \tilde{H}_n(\kappa, x_0') \tag{B.2}$$

is proven after the change of integration variables: $y = \tan(\Phi)$, $z = \tan(\psi - \pi/2 - \Phi)$. Note that the same manipulations give no information on the similar integral between x_0 and x_0'.

The function $g_m[\kappa, x_0]$ of (81) may be expressed as

$$g_m[\kappa, x_0] = 2m\frac{\kappa + l_2}{1+\kappa^2} H_{m+1}(\kappa, x_0) - \left[m - 1 + \frac{m\kappa(\kappa + l_2)}{1+\kappa^2}\right] H_m(\kappa, x_0) + \frac{(\kappa - x_0)^{m-2}}{2m(1+x_0^2)^{m-1}}\left[m - 1 - \frac{2m(\kappa + l_2)(\kappa - x_0)(1+\kappa x_0)}{(1+\kappa^2)(1+x_0^2)}\right]. \tag{B.3}$$

In the same manner, the function $g_m'[\kappa, x_0']$ of (84) may be expressed as

$$g_m'[\kappa, x_0'] = 2m\frac{\kappa + l_2}{1+\kappa^2} \tilde{H}_{m+1}(\kappa, x_0') - \left[m - 1 + \frac{m\kappa(\kappa + l_2)}{1+\kappa^2}\right] \tilde{H}_m(\kappa, x_0') - \frac{(\kappa - x_0')^{m-2}}{2m(1+x_0'^2)^{m-1}}\left[m + 1 - \frac{2m(\kappa + l_2)(\kappa - x_0')(1+\kappa x_0')}{(1+\kappa^2)(1+x_0'^2)}\right]. \tag{B.4}$$

So, using the identity

$$\frac{\kappa - x_0}{1+x_0^2} = \frac{\kappa - x_0'}{1+x_0'^2}$$

resulting from (85), one gets

$$g_m[\kappa, x_0] + g_m'[\kappa, x_0'] = 0.$$

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